25/07/18.

UNIT-II

COMBINATORICS

Mathematical induction:-

Mathematical induction is a technique to prove properties of positive integers.

Let P(n) is a statement involving for all +ve integers n then to prove P(n) is true by mathematical induction principle as follows

i) P(1) is true (Basic step)

ii) Assume P(K) is true
iii) To prove : P(K+1) is true } (Inductive step)

1. Prove by induction $1+2+3+\dots+n = \frac{n(n+1)}{2}$

Sol:

Let $P(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

i) To prove P(1) is true

$n=1,$

$P(1) = 1 = \frac{1(1+1)}{2} = 1$

$\therefore$ P(1) is true.

ii) Assume P(K) is true.

i.e, $P(K) = 1+2+3+\dots+K = \frac{K(K+1)}{2}$ —— ①

iii) To prove : P(K+1) is true,

i.e, To prove : $P(K+1) = \frac{(K+1)((K+1)+1)}{2}$

$P(K+1) = 1+2+3+\dots+K+(K+1)$

$= \frac{K(K+1)}{2} + (K+1)$ ($\because$ by ①)

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$P(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$\therefore$ By mathematical induction,

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

2. Show that $1^3+2^3+3^3+\dots\dots+n^3 = \frac{n^2(n+1)^2}{4}$ by mathematical induction.

Sol:

i) To prove $P(1)$ is true

$n=1$

$P(1) = 1^3 = \frac{1^2(1+1)^2}{4} = 1$

$\therefore P(1)$ is true.

ii) Assume $P(k)$ is true

i.e, $P(k) = 1^3+2^3+\dots+k^3$

$$= \frac{k^2(k+1)^2}{4} \quad \text{--- (1)}$$

iii) To prove $P(k+1)$ is true

i.e, To prove: $P(k+1) = \frac{(k+1)^2((k+1)+1)^2}{4}$

$$P(k+1) = 1^3+2^3+\dots k^3+(k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \because \text{by (1)}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}.$$

$$= \frac{(k+1)^2 [k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2 [k^2+4k+4]}{4}$$

$$= \frac{(k+1)^2 [k+2]^2}{4}$$

$$= \frac{(k+1)^2 ((k+1)+1)^2}{4}$$

$\therefore P(k+1)$ is true

By mathematical induction.

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}.$$

3. Show that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

sol:

$$P(n) = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

i) To prove $P(1)$ is true

$n=1$,

$$P(1) = 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$$

$\therefore P(1)$ is true.

ii) Assume $P(k)$ is true,

i.e. $P(k) = 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ —— ①

iii) To prove : $P(k+1)$ is true

i.e, To prove : $P(k+1) = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$P(k+1) = 1^2+2^2+\dots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \because \text{by } ①$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

$\because 2k+3$
$= 2k+2+1$
$= 2(k+1)+1$

$$= \frac{(k+1)[(k+2)(2k+3)]}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

4. prove by induction the sum of first odd integers is N^2 or show that $1+3+5+\dots+(2n-1)=N^2$, $N \geq 1$

Sol:

$$P(n) = 1+3+\dots+(2n-1) = n^2$$

i) To prove : $P(1)$ is true

$P(1) = 1 = 1^2 = 1$

$\therefore P(1)$ is true

ii) Assume $P(k)$ is true

$$P(k) = 1+3+\dots+(2k-1) = k^2 \quad ①$$

iii) To prove : P(K+1) is true.

To prove : $P(K+1) = (K+1)^2$

$P(K+1) = 1+3+5 \cdots + (2(K+1)-1)$

$= 1+3+5 \cdots + (2K-1) + (2(K+1)-1)$

$= K^2 + 2(K+1) - 1$ ∴ by ①

$= K^2 + 2K + 2 - 1$

$= K^2 + 2K + 1$

$P(K+1) = (K+1)^2$

∴ P(K+1) is true

∴ By mathematical induction,

$1+3+5+ \cdots + 2n-1 = n^2.$

5. Prove that $1+2+2^2+ \cdots + 2^n = 2^{n+1}-1, n \geq 1$

Sol:

Let

$P(n) = 1+2+2^2+ \cdots + 2^n = 2^{n+1}-1$

i) To prove P(1) is true

$P(1) = 1+2 = 2^{1+1}-1 = 3$

∴ P(1) is true.

ii) Assume P(K) is true.

i.e, $P(K) = 1+2+2^2+ \cdots + 2^K = 2^{K+1}-1$

iii) To prove P(K+1) is true.

i.e, To prove $P(K+1) = 2^{(K+1)+1}-1$

$P(K+1) = 1+2+2^2+ \cdots + 2^{K+1}$

$= 1+2+2^2+ \cdots + 2^K + 2^{K+1}$

$= 2^{K+1} - 1 + 2^{K+1}$

$= 2^{K+1}(1+1) - 1$

$= 2(2^{K+1}) - 1 = 2^{K+1+1} - 1$

$P(K+1) = 2^{(K+1)+1} - 1$

By mathematical induction, $1+2+2^2+\dots+2^n = 2^{n+1}-1$

6. Prove that $2+2^2+2^3+\dots+2^n = 2^{n+1}-2$

Sol:

$$P(n) = 2+2^2+2^3+\dots+2^n = 2^{n+1}-2$$

i) To prove $P(1)$ is true:- $P(1) = 2 = 2^{1+1}-2$

$$P(1) = 2^{1+1}-2 = 2$$

$P(1)$ is true.

ii) Assume $P(K)$ is true,

i.e, $P(K) = 2+2^2+2^3+\dots+2^K = 2^{K+1}-2$ —— ①

iii) To prove $P(K+1)$ is true,

i.e, To prove $P(K+1) = 2^{(K+1)+1}-2$

$$P(K+1) = 2+2^2+2^3+\dots+2^{K+1}$$
$$= 2+2^2+2^3+\dots+2^K+2^{K+1}$$
$$= 2^{K+1}-2+2^{K+1}$$
$$= 2^{K+1}(1+1)-2$$
$$= 2(2^{K+1})-2$$
$$= 2^{(K+1)+1}-2$$

By mathematical induction,

$$2+2^2+2^3+\dots+2^n = 2^{n+1}-2$$

Show that $n < 2^n$ (or) $2^n > n$.

Sol:

Let $p(n) = n < 2^n$

i) To prove $p(1)$ is true

$n = 1$

$p(1) = 1 < 2^1$

$\therefore p(1)$ is true.

ii) Assume $p(k)$ is true,

i.e,

$p(k) = k < 2^k$ —— ①

iii) To prove $p(k+1)$ is true,

i.e, to prove $p(k+1) = k+1 < 2^{k+1}$

from ①,

$= k < 2^k$

By adding ① on b/s.

$= k+1 < 2^{k+1}$

$= k+1 < 2^k + 2^k$ $\quad (\because 1 < 2^k \; \forall k)$

$= k+1 < 2^k(1+1)$

$= k+1 < 2^k \cdot 2$

$= k+1 < 2^{k+1}$

$\therefore p(k+1)$ is true.

$\therefore$ By mathematical induction

$n < 2^n$

8. Show that $2^n < n!$ for all $n \geq 4$.

Sol:

$p(n) = 2^n < n!$

i) To prove $p(4)$ is true,

$p(4) = 2^4 < 4!$

$= 16 < 24$

$p(4)$ is true.

ii) Assume $p(k)$ is true.

i.e,

$p(k) = 2^k < k!$ —— ①

iii) To prove $p(k+1)$ is true.

i.e, To prove $p(k+1) = 2^{k+1} < (k+1)!$

from ①,

$2^k < k!$

multiply by 2, on b/s

$2 \cdot 2^k < 2k!$

$2^{k+1} < 2k!$

$2^{k+1} < (k+1)k!$ $\quad$ ($\because 2 < k+1$, for all $k \geq 4$)

$2^{k+1} < (k+1)!$

$\therefore p(k+1)$ is true.

By mathematical induction,

$2^n < n!$

9. Prove that n^3+2n is divisible by 3 using induction method.

Sol:

Let $P(n) = n^3+2n$ is divisible by 3.

i) To prove $P(1)$ is true,

$= 1^3+2(1)$

$P(1) = 3$ is divisible by 3

$\therefore P(1)$ is true.

ii) Assume $P(K)$ is true

i.e, $P(K) = K^3+2K$ is divisible by 3 ——— ①

iii) To prove $P(K+1)$ is true

i.e. To prove $P(K+1) = (K+1)^3+2(K+1)$ is divisible by 3.

$P(K+1) = (K+1)^3+2(K+1)$

$= K^3+1^3+3K^2+3K+2K+2$

$= K^3+3K^2+2K+3K+3$

$= (K^3+2K)+3K^2+3K+3$

$= (K^3+2K)+3(K^2+K+1)$

$\therefore P(K+1)$ is divisible by 3 ($\because$ by ①)

$3(K^2+K+1)$ is multiple of 3. $\therefore P(K+1) = (K+1)^3 + 2(K+1)$ is $\div$ by 3.

By mathematical induction

$P(n) = n^3+2n$ is divisible by 3.

10. Show that $8^n - 3^n$ is multiple of 5.

Sol:

Let $P(n) = 8^n - 3^n$ is $\times$ of 5

i) $p(1)$ is true

$$P(1) = 8^1 - 3^1$$
$$= 5$$

$\therefore P(1)$ is true.

ii) Assume $p(k)$ is true.

i.e,

$$P(k) = 8^k - 3^k \text{ is true}$$
$$8^k - 3^k = 5m$$
$$8^k = 5m + 3^k \quad \text{——} \quad (1)$$

iii) To prove $p(k+1)$ is true.

i.e, $P(k+1) = 8^{k+1} - 3^{k+1}$ is multiple of 5

Consider $8^{k+1} - 3^{k+1}$

$$= 8^k \cdot 8 - 3^k \cdot 3$$
$$= (5m + 3^k) 8 - 3^k \cdot 3$$
$$= 5(8m) + 8 \cdot 3^k - 3 \cdot 3^k$$
$$= 5(8m) + 5 \cdot 3^k$$

$P(k+1) = 5(8m) + 5 \cdot 3^k$ is multiple of 5.

$P(k+1)$ is true.

By mathematical induction,

$8^n - 3^n$ is multiple of 5.

11. Use mathematical induction prove that $H_{2^n} \geq 1+\frac{n}{2}$, where $H_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k}$.

Sol:

Let $P(n) = H_{2^n} \geq 1 + n/2$

i) To prove : $P(0)$ is true.

$$P(0) = H_{2^0} \geq 1 + \frac{0}{2}$$

$$= 1 \geq 1$$

$\therefore P(0)$ is true.

ii) Assume $P(k)$ is true,

$$P(k) = H_{2^k} \geq 1 + \frac{k}{2} \text{ is true} \quad \text{—(1)}$$

iii) To prove $P(k+1)$ is true

$$P(k+1) = H_{2^{k+1}} \geq 1 + \frac{(k+1)}{2} \text{ is true}$$

$$H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{8} + \dots + \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}}$$

($\because$ by diff of H_{2^k})

$$= H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} \dots + \frac{1}{2^{k+1}}$$

$$\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}}$$

$$\geq \left(1 + \frac{k}{2}\right) + 2^k \frac{1}{2^{k+1}}$$

($\because$ There are 2^k terms each $\geq \frac{1}{2^{k+1}}$)

$$\geq \left(1 + \frac{k}{2}\right) + 2^k \frac{1}{2^k \cdot 2}$$

$$\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2}$$

$$\geq \left(1 + \frac{k+1}{2}\right)$$

$\therefore P(k+1)$ is true

By mathematical induction,

$$H_{2^n} \geq 1 + n/2.$$

13. Use mathematical induction. $8^n - 3^n$ is divisible by 5.

Sol:

$P(n) = 8^n - 3^n$ is divisible by 5.

i) $P(1) = 8^1 - 3^1 = 5$ is divisible by 5

$\therefore p(1)$ is true.

ii) Assume $p(k)$ is true

$P(k) = 8^k - 3^k$ is divisible by 5

$8^k - 3^k = 5m$ $\quad \left(\because \frac{8^k - 3^k}{5} = m\right)$

$8^k = 5m + 3^k$ —— ①

iii) To prove $p(k+1)$ is true

$P(k+1) = 8^{k+1} - 3^{k+1}$ is divisible by 5

$P(k+1) = 8^k \cdot 8 - 3^k \cdot 3$

$= (5m + 3^k) \cdot 8 - 3^k \cdot 3$

$= 5(8m) + 8 \cdot 3^k - 3 \cdot 3^k$

$= 5(8m) + 5 \cdot 3^k$

which is multiple of 5

so it is divisible by 5.

$\therefore P(k+1)$ is true

By mathematical induction,

$8^n - 3^n$ is divisible by 5.

PRINCIPLE OF INCLUSION & EXCLUSION:-

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

The principle is used to found the number of elements in the union of two or more sets.

1. If A & B are two sets then the number of elements in their union set $A \cup B$ is given by

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(or)

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

2. If A, B, C are any three sets then.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

3.

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

1. A survey of 500 from a school produced the following information. 200 played volleyball, 120 played hockey: 60 played both volleyball & hockey. How many are not playing either volleyball or hockey.

$$\begin{array}{r} 200 \\ 120 \\ \hline 320 \end{array}$$

$$\begin{array}{r} 500 \\ 320 \\ \hline 220 \end{array}$$

Sol:

Let A denote the students who play Volleyball

B denote the students who play hockey.

Given ~~N~~ $n = 500$

$|A| = 200$

$|B| = 120$

$|A \cap B| = 60$

We know that, by principle of inclusion exclusion, the no of students playing either volleyball or Hockey is

$|A \cup B| = \cancel{260}^{|A|+|B|}\ |A| + |B| - |A \cap B|$

$= 200 + 120 - 60$

$= 260$

∴ The no of students not playing either volleyball or hockey is

$= 500 - 260$

$= 240.$

2. A total of 1232 students have taken a course in telugu, in tamil, 879 have taken a course in hindi. Further 103 and 114 have taken a course in both tamil & telugu, 23 have taken a course in tamil & Hindi and 14 have taken a course in telugu and hindi. If 2092 students have taken atleast one of the tamil, telugu & hindi. How many students have taken a course in all three languages.

Sol:

Let A the students who have taken a course in tamil.

B – Telugu

C = Hindi.

$|A| = 1232$

$|B| = 879$

$|C| = 114$

$|A \cap B| = 103$

$|A \cap C| = 23$

$|B \cap C| = 14$

$|A \cup B \cup C| = 2092$

$$\begin{array}{r} 1232 \\ 879 \\ 114 \\ \hline 2225 \end{array} \qquad \begin{array}{r} 103 \\ 23 \\ 14 \\ \hline 140 \end{array} \qquad \begin{array}{r} 2225 \\ 140 \\ \hline 2085 \end{array}$$

By principle of Inclusion & Exclusion, we have.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |A \cap B \cap C|$$

$$2092 = 2225 - 140 + |A \cap B \cap C|$$

$$2092 = 2085 + |A \cap B \cap C|$$

$$2092 - 2085 = |A \cap B \cap C|$$

$$7 = |A \cap B \cap C|$$

The students taken all the three languages is 7.

3. How many positive integers not exceeding 1000 are divisible by 7 or 11.

Sol:

Let A denote the set of positive integers not exceeding 1000 that are divisible by 7.

Let B denote the set of positive integers not exceeding 1000 that are divisible by 11.

Then,

$$|A| = \left\lfloor \frac{1000}{7} \right\rfloor$$

$$= \lfloor 142.85 \rfloor$$

$$= 142.$$

$$|B| = \left\lfloor \frac{1000}{11} \right\rfloor$$

$$= \lfloor 90.90 \rfloor$$

$$= 90$$

$$|A \cap B| = \left\lfloor \frac{1000}{7 \times 11} \right\rfloor$$

$$= \left\lfloor \frac{1000}{77} \right\rfloor$$

$$= \lfloor 12.98 \rfloor$$

$$= 12.$$

The number of positive integer not exceeding 1000 that are divisible either 7 or 11 is

$|A \cup B|$ by principle of inclusion exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 142 + 90 - 12$$

$$= 220$$

There are 220 positive integer not exceeding 1000 divisible by either 7 or 11.

4) Determine n such that $1 \le n \le 100$ which are not divisible by 5 or by 7.

Sol:

Let A denote the number $1 \le n \le 100$ are divisible by 5.

Let B denote the number $1 \le n \le 100$ are divisible by 7.

Then,

$$|A| = \left\lfloor \frac{100}{5} \right\rfloor$$

$$= 20$$

$$|B| = \left\lfloor \frac{100}{7} \right\rfloor$$

$$= 14.2$$

$$= 14$$

$$|A \cap B| = \left\lfloor \frac{100}{5 \times 7} \right\rfloor$$

$$= \frac{100}{35}$$

$$= \lfloor 2.8 \rfloor$$

$$= 2$$

The number $1 \le n \le 100$ which are not divisible by 5 or 7 is $|A \cup B|$ by principle of inclusion exclusion.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 20 + 14 - 2$$

$$= 32$$

$$n - |A \cup B|$$

$$= 100 - 32$$

$$= 68$$

There are 68 number that are not divisible by 5 or by 7.

Q. Find the no. of integers between 1 to 100 that are divisible by

i) 2, 3, 5 or 7

ii) 2, 3, 5 but not by 7.

Sol:-

i) Let A ~~denote~~, B, C, D denote the no. of integers between 1 to 100 that are divisible by 2, 3, 5 or 7. respectively.

$$|A| = \left\lfloor \frac{100}{2} \right\rfloor$$

$$= 50$$

$$|B| = \frac{100}{3}$$

$$= 33$$

$$|C| = \frac{100}{5}$$

$$= 20$$

$$|D| = \frac{100}{7}$$

$$= 14.$$

$$|A \cap B| = \left\lfloor \frac{100}{2 \times 3} \right\rfloor$$

$$= \frac{100}{6}$$

$$= 16$$

$$|A \cap C| = \left\lfloor \frac{100}{2 \times 5} \right\rfloor$$

$$= \frac{100}{10}$$

$$= 10$$

$$|A \cap D| = \left\lfloor \frac{100}{2 \times 7} \right\rfloor$$

$$= \frac{100}{14} = 7.$$

$|B\cap C| = \left\lfloor \dfrac{100}{3\times5} \right\rfloor$

$= \dfrac{100}{15} = 6$

$|B\cap D| = \left\lfloor \dfrac{100}{3\times7} \right\rfloor$

$= \dfrac{100}{21}$

$= 4$

$|A\cap B\cap C| = \left\lfloor \dfrac{100}{2\times3\times5} \right\rfloor$

$= \dfrac{100}{30}$

$= 3$

6x7

$|A\cap B\cap D| = \left\lfloor \dfrac{100}{2\times3\times7} \right\rfloor$

$= \dfrac{100}{42}$

$= 2$

$|A\cap C\cap D| = \left\lfloor \dfrac{100}{2\times5\times\cancel{10}7} \right\rfloor$

$= \dfrac{100}{\cancel{10}70}$

$= 1$

$|B\cap C\cap D| = \left\lfloor \dfrac{100}{3\times5\times7} \right\rfloor$

$= \dfrac{100}{105}$

$= 0$

$|A\cap B\cap C\cap D| = \left\lfloor \dfrac{100}{2\times3\times5\times7} \right\rfloor$

$= \dfrac{100}{210}$

$= 0$

By principle of Inclusion exclusion,

$$|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D|-|A\cap B|$$
$$-|A\cap C|-|C\cap D|-|A\cap D|-|B\cap C|-|B\cap D|$$
$$+|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D|+$$
$$|B\cap C\cap D|-|A\cap B\cap C\cap D|$$

$$= 50+33+20+14-16-10-\overset{2}{\cancel{7}}-6-4+3+2$$
$$+1+0+0$$

$$= 78.$$

$\therefore$ 78 no. of positive integers betwen 1 to 100 are divisible by 2, 3, 5 or 7.

$100-78 = 22$

ii) The no. of integers 1 to 100 are divisibly 2, 3, 5 but not by 7.

$$= |A\cap B\cap C| - |A\cap B\cap C\cap D|$$
$$= 3-0$$
$$= 3$$

5. How many Integers between 1 to 100 that are

i) not divisible by 7, 11 or 13.

ii) divisible by 3 but not by 7.

Sol:

Let A, B, C donote the no. of integers between 1 to 100 that are divisible by 7, 11 or 13 respectively.

Now,

$$|A| = \frac{100}{7}$$
$$= 14.$$

$$|B| = \frac{100}{11}$$

$$= 9$$

$$|C| = \frac{100}{13}$$

$$= 7.$$

$$|A \cap B| = \left\lfloor \frac{100}{7 \times 11} \right\rfloor$$

$$= \frac{100}{77}$$

$$= 1$$

$$|A \cap C| = \left\lfloor \frac{100}{91} \right\rfloor$$

$$= 1$$

$$|B \cap C| = \left\lfloor \frac{100}{11 \times 13} \right\rfloor$$

$$= \frac{100}{143}$$

$$= 0$$

$$|A \cap B \cap C| = \frac{100}{7 \times 11 \times 13}.$$

$$= \frac{100}{1001}$$

$$= 0$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 14 + 9 + 7 - 1 - 1$$

$$= 28.$$

The no of integers not divisible by 7, 11, 13. $= 100 - 28 = 72.$

72 positive integer is not divisible by 7, 11, or 13.

ii) Let A, B denote the no between 1 to 100 that are divisible by 3 or 7.

$|A| = \frac{100}{3}$

$= 33$

$|B| = \frac{100}{7}$

$= 14$

$|A \cap B| = \left\lfloor \frac{100}{21} \right\rfloor$

$= 4.$

The no of integers divisible by 3 but not 7,

$= |A| - |A \cap B|$

$= 33 - 4$

$= 29.$

29 no. of integers between 1-100 is not divisible by 7.

RECURRENCE RELATION:-

An equation that expresses a_n; the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence namely $a_0, a_1, a_2, \ldots\ldots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a positive integer is called a recurrence relation for $\{a_n\}$ or difference equation.

Basic of Computing :-

Two basic of Counting principles are.

1. product rule.
2. Sum rule.

Product rule :-

If there are n_1 ways to do ~~way~~ the first task and n_2 ways to do the second task after the first task has been done.

Then there are $n_1 \cdot n_2$ ways to do both the task.

1. How many different bit strings are there of length 7

Sol:

Each of the 7 bit can be choosen in two ways since each bit is either 0 or 1.

By product rule there are

0,1	0,1	0,1	0,1	0,1	0,1	0,1

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$$

$$= 128$$

128 different bit strings of length 7.

2. How many different ~~ee~~ 8-bit strings are there that begin and end with 1.

Sol:

1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	1

~~A~~ 8-bit string that begin and end with 1 can be constructed in 6 steps.

Each bit can be selected in two ways.

i.e., The total no. of 8-bit strings that begin and end with 1.

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$$
$$= 64$$

3. How many positivity test can be formed using the digits 3, 4, 4, 5, 5, 6, 7. If n has to exceed 50,00000.

Sol:

The Integer 5, 6, 7 are used to fix the first place.

5, 6, 7 (3)	x (7)	x (7)	x (7)	x (7)	x (7)	x (7)

The first place can be filled in three different ways the remaining 6 digits mark x. and each ~~ea~~ digit x can be filled ~~and~~ 7 different ways.

i.e. the no. of +ve integer exceed 5000000~~e~~

$$= 3 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$$

$$= 352,947.$$

Sum rule:-

If a first task can be done in n_1 ways and a second task in n_2 ways. And if this task cannot be done at the sametime, then there are n_1+n_2 ways to do one of this task.

1. A student can choose the computer project from one of the three task. If three task possible project respectively contain 23, 15 and 19. How many projects are there to choosen from,

sol:

The students can choose a project from the first in 23 ways, from the second in 15 ways and from the third in 19 ways. Hence there are

$23+15+19 = 57$.

57 projects to choose them.

PIGEON HOLE PRINCIPLE:-

If $n+1$ pigeon occupies n holes then atleast one hole has more than one pigeon.

ex:

Among any group of 367 boys there must be atleast two with the same birthday because there are only 366 possible birth day.

Generalized pigeon hole principle:- $(m>n)$

If m pigeon occupies n holes then atleast one hole has more than $\left[\frac{m-1}{n}\right]+1$ pigeon.

Show that 100 people atleast 9 of them where born in same month.

Sol:

Here no. of pigeon = m = no. of people = 100

no. of holes = n = no. of month = 12

Then by G.P.H.P

$$\left[\frac{m-1}{n}\right]+1$$

$$= \left[\frac{100-1}{12}\right]+1$$

$$= \left[\frac{99}{12}\right]+1$$

$$= 8.25+1$$

$$= 9.25$$

$$= 9$$

9 were born in same month.

Show that if 7 colours are used to paint 50 bicycles, atleast 8 bicycles will be the same colour.

Sol:

$$= \left[\frac{50-1}{7}\right]+1$$

$\therefore m = 50$ - bicycles ← pigeons
$n = 7$ ← colours - holes.

$$= [49/7]+1$$

$$= 7+1$$

$$= 8.$$

Hence Proved.

1\. Show that if 25 dictionaries in a library contain a total of 40,325 pages, then one of the dictionaries must have atleast 1614 pages.

Sol:

$m = 40325$: no. of pages = pigeon

$n = 25$: no. of dictionaries = holes.

By pigeon hole principle,

$= \left\lfloor \frac{m-1}{n} \right\rfloor + 1$

$= \left\lfloor \frac{40325-1}{25} \right\rfloor + 1$

$= \left\lfloor \frac{40324}{25} \right\rfloor + 1$

$= 1612.96 + 1$

$= 1613.96$

$= 1614$ pages.

Find the minimum no. of students need to guarantee that 5 of them belong to same subject, having major as English, maths, Physics & chemistry.

Sol.

Let no. of Students = m = no. of pigeon = ?

no. of Subject = n = no. of holes = 4

Here

By generalized pigeon hole principle,

$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 5$

$\left\lfloor \frac{m-1}{4} \right\rfloor + 1 = 5$

$\left\lfloor \frac{17-1}{4} \right\rfloor + 1$

$\frac{16}{4} + 1$

$4 + 1$

5

$$\frac{m-1+4}{4} = 5$$

$$m+8 = 20$$

$$\boxed{m = 17}$$

How many people must you have to guarantee that atleast 9 of them will have birthday in the same day of the week.

Sol:

Let no. of people = m = no. of pigeon = ?

no. of days = n = no. of holes = 7.

By generalized pigeon hole principle,

$$\left\lfloor \frac{m-1}{7} \right\rfloor + 1 = 9$$

$$\frac{m-1+7}{7} = 9$$

$$m+6 = 63$$

$$m = 69-6$$

$$\boxed{m = 57}$$

PERMUTATION:-

A permutation is an arrangement of a number of objects in a definite order taken some or all at a time.

A no. of permutations of 'n' different things taken 'r' at a time is denoted by the symbol nPr (or) $P(n,r)$

Formula:

$$nPr = \frac{n!}{(n-r)!}$$

5. How many people must you have to guarantee that atleast 9 of them will have birthday in the same day of the week.

Soln: let people $= m =$ no. of pigeon

day of week $= n =$ no. of holes $= 7$

$$\frac{m-1}{n}+1=9$$

$$\frac{m-7}{7}+1=9 \qquad \boxed{m=57}$$

PERMUTATION

Permutation :- A permutation is an arrangement of a number of objects in a definite order, taken some or all at a time.

The no. of permutations of 'n' different things taken 'r' at a time is denoted by the symbol nP_r or $P(n,r)$

Formula :- $nP_r = \dfrac{n!}{(n-r)!}$

Note :- The number of permutations of n things taken all at a time is $n!$ ie, $nP_n = n!$

1) In how many different ways can the letters of the word "HEXAGON" be permuted.

Soln: The word HEXAGON has 7 different letters, which can be arranged among themselves in

$$P(7,7) = 7P_7 = \frac{7!}{(7-7)!} = 7! = 5040 \text{ ways.}$$

2) Find the no. of ways in which the letters of the word "TRIANGLE" can be arranged such that (i) vowels occur together (ii) vowels occupy odd places

Soln The word TRIANGLE has 8 letters of which 3 are vowels (I, A, E)

(i) Imagine the 3 vowels written on a single plate and the 5 consonants on 5 different plates. These 6 plates can be arranged in $6P_6 = 6! = 720$ ways

The three vowels on a single plate can be arranged among themselves in $3P_3 = 3! = 6$ ways

$\therefore$ The required no. of words $= 720 \times 6 = 4320$.

(ii) The four odd places (1st, 3rd, 5th, 7th) can be filled by the 3 vowels in $4P_3 = 4 \times 3 \times 2 = 24$ ways

The remaining 5 places can be filled by the 5 consonants in $5P_5 = 5! = 120$ ways. The required no. of words $= 24 \times 120 = 2880$.

3) In how many ways 7 women and 3 men be arranged in a row, if three men must always stand next to each other?

Soln $\underbrace{MMM}_{x}$ w w w w w w w

The three men can be arranged among themselves in $3! = 6$ ways

Taking 3 men standing together as a single object x and the 7 women as other 7 objects, we have to arrange 8 objects taken all at a time. There are 8! ways of arranging the 8 objects

Hence the no. of ways of arranging 7 women and 3 men with the condition that the 3 men must always stand next to each other

$= 8!\,3! = 241920.$

4) How many permutations can be made out of the letters of the word "COMPUTER" How many of these (i) begin with C (ii) end with R (iii) Begin with C and end with R (iv) C and R occupy the end places.

Soln :- The word COMPUTER has 8 distinct letters.

$\therefore$ No. of permutations of the word COMPUTER

$= 8! = 40320.$

(i) The word that begin with C is C x x x x x x x. The remaining 7 letters can be arranged among themselves in 7! ways. $\therefore$ The total of permutations starting with C $= 7! = 5040$

(ii) The word that end with R is of the form x x x x x x x R. After fixing R in the end, the remaining 7 positions can be filled in 7! different ways. $\therefore$ No. of permutation ending with R $= 7! = 5040.$

(iii) The 1st and last positions are filled with C and R and the remaining 6 position can filled in 6! ways C x x x x x x R

$\therefore$ The no. of permutations begin with C and end with R $= 6! = 720$

(iv) C and R occupy the end positions in $2! = 2$ ways.

x x x x x x C R, x x x x x x R C The 1st 6 positions can arranged in 6! ways. $\therefore$ The total no. of permutations in which C and R occupy the end positions $= 2! \times 6! = 1440.$

Problems based on repetation:-

1) In how many different ways can the letters of the word "ALLAHABAD" be permuted.

Soln: The word "ALLAHABAD" has 9 letters in all.

The letters A appears 4 times, the letter L appears 2 times and the remaining 3 letters H, B, D appear once.

∴ The required no. of permutations $= \frac{9!}{4!\,2!\,1!\,1!} = 7560$

2) How many permutations are there in the word "MISSISSIPPI" Ans: $\frac{11!}{4!\,4!\,2!} = 34650.$

3. what is the no. of arrangements of all the six letters in the word PEPPER. Ans: $\frac{6!}{3!\,2!} = 60$

4. How many different words are there in the word "ENGINEERING" $\frac{11!}{3!\,3!\,2!\,2!} = 2,77,200$

5) How many different signals can be transmitted by arranging 3 red, 2 yellow and 2 green flags on a pole. (Assume that all the 7 flags are used to transmit a signal).

Soln: Required no. of signals $= \frac{7!}{3!\,2!\,2!} = 210.$

6) There are 5 red, 4 white and 3 blue marbles in a bag. They are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the no. of different arrangements. $\frac{12!}{5!\,4!\,3!} = 27720.$

Combination :- A combination is a selection of objects without regard to order.

The no. of Combinations of n different things taken r at a time is denoted by nC_r (or) $c(n,r)$ (or) $\binom{n}{r}$

Formula :- $nC_r = \frac{n!}{r!(n-r)!}$

Note: $nC_n = 1$, $nC_r = nC_{n-r}$, $0! = 1$

1) How many 16-bit strings are there containing exactly 5 zeros?

Soln: The no. of 16-bit strings with exactly 5 0's

$= 16C_5 = 4368.$

2) A Committee of 5 is to be selected from 6 boys and 5 girls. Determine the no. of ways of selecting the committee if it is to consist of atleast 1 boy and 1 girl.

Soln The Committee may consist of

(i) 1 boy, 4 girl, 2) 2 boys 3 girl, 3) 3 boys 2 girl, 4) 4 boys 1 girl

$= 6C_1 \times 5C_4 = 30$, $6C_2 \times 5C_3 = 150$, $6C_3 \times 5C_2 = 200$, $6C_4 \times 5C_1 = 75$

3) There are 6 white marbles and 5 black marbles in a bag. Find the no. of ways of drawing 4 marbles from the bag if 1) They can be of any colour 2) 2 must be white and 2 must be black 3) They must all be of the same colour.

Soln Total no. of marbles in the bag $= 6+5 = 11$

(i) No. of ways of drawing 4 marbles of any colour

$= 11C_4 = 330.$

(ii) No. of ways of drawing 2 white and 2 black marbles $= 6C_2 \times 5C_2 = 150$

(iii) No. of ways of drawing 4 white marbles $= 6C_4 = 15$

No. of ways of drawing 4 black marbles $= 5C_4 = 5$.

$\therefore$ No. of ways of drawing 4 marbles of same colour $= 15 + 5$
$= 20$.

---

4) From a club consisting of 6 men and 7 women, in how many ways can we select a committee of (a) 3 men and 4 women (b) 4 persons which has atleast one women (c) 4 persons that has atmost one man (d) 4 persons that has persons of both sexes (e) 4 persons so that two specific members are not included?

Soln:- (a) 3 men and 4 women can be selected from 6 men in $6C_3$, 4 women can be selected from 7 W. in $7C_4$

$\therefore$ The committee of 3 m & 4 W can be selected in $= 6C_3 \times 7C_4 = 700$

(b) 4 persons which has atleast one women, we have to select 3 men and 1 women (or) 2 men and 2 women or 1 men and 3 women or no man and 4 women.

$$= 6C_3 \times 7C_1 + 6C_2 \times 7C_2 + 6C_1 \times 7C_3 + 6C_0 \times 7C_4$$

$$= 140 + 315 + 210 + 35 = 700$$

(c) 4 persons that has atmost one man, we have to select no men and 4 women or 1 men and 3 women.

$$= 6C_0 \times 7C_4 + 6C_1 \times 7C_3 = 245$$

(d) For the committee to have persons of both sexes, the selection must include 1 man and 3 women or 2 men and 2 women or 3 men and 1 women. This selection can be done in

$$= 6C_1 \times 7C_3 + 6C_2 \times 7C_2 + 6C_3 \times 7C_1$$

$$= 210 + 315 + 140 = 665.$$

e) After removing the two specific members, 2 members can be selected from the remaining in ${}^{11}C_2$ ways.

$\therefore$ The no. of selections not including these 2 members

$= {}^{13}C_4 - {}^{11}C_2 = 715 - 55 = 660.$

<u>Combinations</u> <u>with Repetitions</u>:- The no. of r-combinations of n kinds of objects, if repetitions of the objects is allowed $= C(n+r-1, r)$. (or) ${}^{n+r-1}C_r$

1) How many solutions does the eqn $x_1 + x_2 + x_3 = 11$ have where $x_1, x_2, x_3 \geq 0$ and are integers.

<u>Soln</u>: here $n = 3$, $r = 11$

hence the no. of soln is $= {}^{n+r-1}C_r$

$= {}^{3+11-1}C_{11} = 78$

2) find the no. of soln $x_1 + x_2 + x_3 = 11$, with $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$.

<u>Soln</u> $n = 3$, $r = 11 - 1 - 2 - 3 = 5$

The no. of soln is $= {}^{n+r-1}C_r = {}^{3+5-1}C_5 = 21.$

RECURRENCE RELATION:-

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a positive integer is called a recurrence relation for $\{a_n\}$ or difference equation.

For example,

$$a_{n+2} = a_{n+1} + 2a_n$$

Formation of recurrence relation :-

1. Form the recurrence relation from $S(k) = 5(2^k)$, $k > 0$

Sol:

Given:

$$S(k) = 5(2^k)$$

$$S(k-1) = 5(2^{k-1})$$

$$S(k-1) = 5\, 2^k \cdot 2^{-1}$$

$$S(k-1) = 5\frac{(2^k)}{2}$$

$$S(k-1) = \frac{S(k)}{2}$$

$$2(S(k-1)) = S(k)$$

with initial value $S(0) = 5$ → same.

$S(k+1) = 5\, 2^{k+1}$

$= 2 \cdot 5(2^k)$

$S(k+1) - 2S(k) = 0$

2. Find the recurrence relation $S(k) = 5(2^k)$, $k \geq 0$

Find the recurrence relation from $y_n = A\, 2^n + B(-3)^n$

Sol.

Given:

$$y_n = A\, 2^n + B(-3)^n \quad \text{—— (1)}$$

$$y_{n+1} = A 2^{n+1} + B(-3)^{n+1}$$

$$y_{n+1} = A\, 2^n \cdot 2 + B(-3)^n \cdot (-3)$$

$$y_{n+1} = 2A 2^n - 3B(-3)^n \quad \text{—— (2)}$$

$$y_{n+2} = A\, 2^{n+2} + B(-3)^{n+2}$$

$$y_{n+2} = A\, 2^n \cdot 4 + B(-3)^n \cdot (-3)^2$$

-3×-3.

$$y_{n+2} = 4A 2^n + 9B(-3)^n \quad \text{—— (3)}$$

Eliminate $A2^n$, $B(-3)^n$ from above 3 equations.

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$$y_n(18+12) - 1(9y_{n+1} + 3y_{n+2}) + 1(4y_{n+1} - 2y_{n+2}) = 0$$

$$= y_n(30) - 9y_{n+1} - 3y_{n+2} + 4y_{n+1} - 2y_{n+2} = 0$$

$$30y_n - 5y_{n+1} - 5y_{n+2} = 0$$

$\div 5$

$$6y_n - y_{n+1} - y_{n+2} = 0$$

General form of K^{th} order reccurence relation.

$$C_0 y_{n+k} + C_1 y_{n+k-1} + C_2 y_{n+k-2} \ldots\ldots + C_n y_n = f(n)$$

is the general form of the K^{th} order reccurence relation. where $C_0, C_1, C_2 \ldots\ldots C_n$ are constants and $f(n)$ is a function of n.

This recurrence relation is non-homogeneous type. when $f(n) = 0$ (RHS = 0) the R.R is homogeneous type.

Results:-

$F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ is called Fibonacci series recurence relation.

For the sequence of numbers $\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$ and the initial conditions $F_0 = 0$ and $F_1 = 1$

Solution of recurrence relation:-

Consider the recurrence relation $c_0 y_{n+2} + c_1 y_{n+1} + c_n y_n = f(n)$

$\therefore$ The solution is

$$\boxed{y_n = H.S + P.S}$$

where,

H.S = Homogeneous solution.

P.S = particular solution.

Rules to find H.S:-

Step 1 : Find the characteristic equation $c_0 x^2 + c_1 x + c_2 = 0$

Step 2 : solve the C.E and we get the roots

Step 3 : If α_1, α_2 are the roots then,

1. $H.S = c_1 \alpha_1^n + c_2 \alpha_2^n$, if the roots are distinct

2. $H.S = (c_1 + n c_2) \alpha^n$, if the roots are same $(\alpha_1 = \alpha_2)$

3. $H.S = (c_1 \cos n\theta + c_2 \sin n\theta) r^n$, if the roots are imaginary.

Rules to find P.S:-

Form of $f(n)$	General form to be Assumed.
K (a constant)	A
K^n (K is a constant)	AK^n

K^n (K is a root of characteristic equation)	AnK^n
K^n (K is a double root of characteristic equation)	An^2K^n
$f(n)$ (a polynomial of degree 2)	$A_0 + A_1 n + A_2 n^2$
$f(n)$ (a polynomial of degree r)	$A_0 + A_1 n + A_2 n^2 \ldots\ldots + A_r n^2$
$K^n f(n)$, ($f(n)$ is a polynomial of degree r & K is constant	$[A_0 + A_1 n + A_2 n^2 \ldots + A_r n^r] K^n$

Problems based on Homogeneous type:-

1. Solve the recurrence relation $y_n - 7y_{n-1} + 10y_{n-2} = 0$ satisfying the condition $y_0 = 0$ and $y_1 = 6$.

Sol:

Given,

$$y_n - 7y_{n-1} + 10y_{n-2} = 0$$

The characteristic equation is,

$$x^2 - 7x + 10 = 0$$

$$(x-5)(x-2) = 0$$

$$x = 2, 5$$

$$\therefore H.S = C_1\alpha_1^n + C_2\alpha_2^n$$

$$= C_1 2^n + C_2 5^n$$

$$\therefore y_n = H.S = C_1 2^n + C_2 5^n \quad \text{—— (1)}$$

Given

$y_0 = 0$, put $n = 0$ in (1)

y_0: $0 = c_1 2^0 + c_2 5^0$

$\Rightarrow c_1 + c_2 = 0$ —— (2)

$y_1 = 6$ put $n = 1$ in (1)

$y_1 = c_1 2^1 + c_2 5^1$

$6 = c_1 \cdot 2 + c_2 5$

$2c_1 + 5c_2 = 6$ —— (3)

To solve 2 & (3)

$$c_1 + c_2 = 0$$
$$2c_1 + 5c_2 = 6$$
$$+ -5$$
$$c_1 - 21c_2 = 6$$

(2) $\times 2 \Rightarrow$ $2c_1 + 2c_2 = 0$

(3) $\times 1 \Rightarrow$ $2c_1 + 5c_2 = 6$

$$-3c_2 = -6$$

$$c_2 = \frac{-6}{-3}$$

$$\boxed{c_2 = 2}$$

c_2 in (1)

$$2c_1 + 2(2) = 0$$

$$2c_1 + 4 = 0$$

$$2c_1 = -4$$

$$c_1 = -4/2$$

$$\boxed{c_1 = -2}$$

from (1),

$$y_n = -2(2^n) + 2(5^n)$$

2. Find an explicit formula for the fibonacci sequence or find a recurrence relation for the fibonacci sequence of numbers and obtain its solution.

Sol:

W.K.T. the fibonacci sequence $0, 1, 2, 3, 5, \ldots$ satisfying the recurrence relation.

$$F_n = F_{n-1} + F_{n-2}$$

$$F_n - F_{n-1} - F_{n-2} = 0$$

and also satisfying the initial conditions

$$F(0) = 0, \; F_1 = 1$$

Now the characteristic equation is

$$x^2 - x - 1 = 0$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1+\sqrt{5}}{2}, \; \frac{1-\sqrt{5}}{2}$$

$$\therefore H.S = C_1 \alpha_1^n + C_2 \alpha_2^n$$

$$F_n = H.S = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{--- (1)}$$

Given

$F_0 = 0$, put $n=0$ in (1)

$$0 = C_1 + C_2 \quad \text{—— (2)}$$

Given:

$F_1 = 1$ put $n=1$ in (1)

$$1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{—— (3)}$$

Solve (2) & (3)

$$(2) \times \left(\frac{1+\sqrt{5}}{2}\right) : \left(\frac{1+\sqrt{5}}{2}\right)C_1 + \left(\frac{1+\sqrt{5}}{2}\right)C_2 = 0$$

$$(3) \qquad : \cancel{\left(\frac{1+\sqrt{5}}{2}\right)C_1} + \left(\frac{1-\sqrt{5}}{2}\right)C_2 = 1$$

$$\left(\frac{1+\sqrt{5}}{2}\right)C_2 - \left(\frac{1-\sqrt{5}}{2}\right)C_2 = -1$$

$$\cancel{\frac{C_2}{2}} + \frac{\sqrt{5}}{2}C_2 - \cancel{\frac{C_2}{2}} + \frac{\sqrt{5}}{2}C_2 = -1$$

$$\frac{2\sqrt{5}}{2}C_2 = -1$$

$$\sqrt{5}\,C_2 = -1$$

$$\boxed{C_2 = -1/\sqrt{5}}$$

subs

$C_2 = \frac{-1}{\sqrt{5}}$ in (2)

$$C_1 - \frac{1}{\sqrt{5}} = 0$$

$$\Rightarrow C_1 = 1/\sqrt{5}$$

$$\therefore F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 0$ with $a_0 = 1, a_1 = 4$.

Sol:

Given:

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x = 3, 3$$

$\therefore$ $H.S = (C_1 + n C_2)\alpha^n$

$\therefore$ $a_n = (C_1 + n C_2) 3^n$

$$a_n = C_1 3^n + C_2 n 3^n \quad \text{---} (1)$$

Given: $a_0 = 1$ put $n = 0$ in (1)

$$1 = C_1 + C_2(0)$$

$$\boxed{C_1 = 1}$$

$a_1 = 4$, put $n = 1$ in (1)

$$4 = C_1 3^1 + C_2 (1) 3^1$$

$$4 = 3C_1 + 3C_2$$

$$4 = 3(1) + 3C_2$$

$$4 - 3 = 3C_2$$

$$1 = 3C_2$$

$$\boxed{C_2 = 1/3}$$

Subs C_1, C_2 in (1)

$$a_n = 1(3^n) + 1/3 \, n 3^n$$

$$a_n = 3^n (1 + n/3)$$

Problems based on non-homogeneous type:-

1) Solve the recurrence relation $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ subject to condition $y_0 = 0$ and $y_1 = 2$.

Sol:

Given:

$$y_{n+2} - 5y_{n+1} + 6y_n = 5^n \quad \text{—— (1)}$$

The characteristic equation

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$x = 2, 3$$

$$\therefore H.S = C_1 2^n + C_2 3^n \quad \text{—— (2)}$$

Assume the P.S as $y_n^{(P)} = A5^n$

subs $y_n = A5^n$ in the given r.r.

$$A5^{n+2} - 5A5^{n+1} + 6A5^n = 5^n$$

$$\cancel{A5^{n+2}} - \cancel{A5^{n+2}} + 6A5^n = 5^n$$

$$6A5^n = 5^n$$

$$6A = 1$$

$$\boxed{A = 1/6}$$

$$\therefore y_n^{(P)} = 1/6\, 5^n \quad \text{—— (3)}$$

$$\therefore y_n = y_n^{(H)} + y_n^{(P)}$$

$$y_n = C_1 2^n + C_2 3^n + 1/6\, 5^n \quad \therefore \text{by (2) \& (3)}$$

Given

$y_0 = 0, n = 0$

$$0 = C_1 + C_2 + 1/6$$

$$\Rightarrow C_1 + C_2 = -1/6 \quad \text{—— (4)}$$

Given:

$y_1 = 2, \ n = 1$

$2 = C_1(2)^1 + C_2(3)^1 + \frac{1}{6} 5^1$

$2 = 2C_1 + 3C_2 + 5/6$

$2C_1 + 3C_2 = 2 - 5/6$

$2C_1 + 3C_2 = 7/6$ — (5)

To solve (4) & (5)

(4) × 2 : $2C_1 + 2C_2 = -2/6$

(5) × 1 : $2C_1 + 3C_2 = 7/6$

$-C_2 = -9/6$

$\boxed{C_2 = 3/2}$

Sub $C_2 = 3/2$ in (4)

$C_1 + 3/2 = -1/6$

$C_1 = -1/6 - 3/2$

$= \frac{-1-9}{6}$

$= \frac{-10}{6}$

$\boxed{C_1 = -5/3}$

$\therefore y_n = -5/3 (2)^n + 3/2 (3)^n + 1/6 \, 5^n$

Solve the D.E $y_{n+2} - 6y_{n+1} + 8y_n = 3n+5$

Sol:

Given: $y_{n+2} - 6y_{n+1} + 8y_n = 3n+5$ — (1)

The C.F is,

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$x = 2, 4$

$\therefore$ H.S $= C_1 2^n + C_2 4^n$ — (2)

Assume the P.S as $y_n^{(P)} = A + Bn$ $A = 5$ $3n = Bn$

subs $y_n = A + Bn$ in ①,

$A + B(n+2) - 6[A + B(n+1)] + 8[A + Bn] = 3n+5$

$A + Bn + 2B - 6A - 6Bn - 6B + 8A + 8Bn = 3n+5$

$3A - 4B + 3Bn = 3n + 5$

Equating the coefficient of 'n'

$3B = 3$

$\boxed{B = 1}$

Equating the coefficient of constant.

$3A - 4B = 5$

$3A - 4(1) = 5$

$3A - 4 = 5$

$3A = 5 + 4$

$A = 9/3$

$\boxed{A = 3}$

$\therefore y_n^{(P)} = 3 + n$

$\therefore y_n = y_n^{(H)} + y_n^{(P)}$

$= C_1 2^n + C_2 4^n + 3 + n$

Solve the r.r $G(k) - 7G(k-1) + 10G(k-2) = 6 + 8k$.

Sol:

Given:

$G(k) - 7G(k-1) + 10G(k-2) = 6 + 8k$. —— ①

The C.E is,

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0$

$x = 2, 5$

$$H.S = C_1 2^n + C_2 5^n \quad \text{——— (2).}$$

Assume P.S as $G_k^{(P)} = A + Bk$

Sub. $G_k = A + Bk$ in (1)

$$(A+Bk) - 7[A + B(k-1)] + 10[A + B(k-2)] = 6 + 8k.$$

$$A + Bk - 7A - 7Bk + 7B + 10A + 10Bk - 20B = 6 + 8k.$$

$$4A - 13B + 4Bk = 6 + 8k.$$

Equating the coefficient of 'k'

$$4B = 8$$

$$\boxed{B = 2}$$

Equating the coefficient of constant

$$4A - 13B = 6$$

$$4A - 13(2) = 6$$

$$4A - 26 = 6$$

$$4A = 6 + 26$$

$$4A = 32$$

$$A = 32/4$$

$$\boxed{A = 8}$$

$$G_k^{(P)} = 8 + 2k$$

$$\therefore G_k = G_k^{(H)} + G_k^{(P)}$$

$$= C_1 2^n + C_2 5^n + 8 + 2k.$$

Solve the r.r $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$

where $n \geq 0$, $a_0 = 1$, $a_1 = 4$.

Sol:

Given:

$$a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$$

The C.E is,

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x = 3, 3$$

$$H.S = (C_1 + nC_2)\alpha^n$$

$$= (C_1 + nC_2)3^n$$

$$= C_1 3^n + C_2 n 3^n$$

Assume the P.S as $a_n^{(p)} = A2^n + Bn^2 3^n$

subs $a_n = A2^n + Bn^2 3^n$ in (1)

$$A2^{n+2} + B(n+2)^2 3^{n+2} - 6[A2^{n+1} + B(n+1)^2 3^{(n+1)}] + 9[A2^n + Bn^2 3^n] = 3(2^n) + 7(3^n)$$

$$A2^n \cdot 2^2 + B(n^2+4+4n)3^n \cdot 3^2 - 6[A2^n \cdot 2 + B(n^2+1+2n) 3^n \cdot 3] + 9[A2^n + Bn^2 3^n] = 3(2^n) + 7(3^n)$$

$4A2^n + 9Bn^2 3^n + 36B3^n + 36Bn3^n - 12A2^n$
$- 18Bn^2 3^n - 18B3^n - 36Bn3^n + 9A2^n$
$+ 9Bn^2 3^n = 3(2^n) + 7(3^n)$

equating the coeffi of '2^n'

$4A - 12A + 9A = 3$

$\boxed{A = 3}$

Equating the coeffi of '3^n'

$36B - 18B = 7$

$18B = 7$

$\boxed{B = 7/18}$

$\frac{36 \; 18}{18}$

$\therefore a_n^{(P)} = 3 2^n + 7/18 \, n^2 3^n$

$\therefore a_n = a_n^{(H)} + a_n^{(P)}$

$= C_1 3^n + C_2 n 3^n + 3 2^n + 7/18 \, n^2 3^n$

$= (C_1 + C_2 n) 3^n + 3 2^n + 7/18 \, n^2 3^n$

$= C_1 3^n + C_2 n 3^n + 3(2^n) + \frac{7}{2(3^2)} n^2 3^n$

$= C_1 3^n + C_2 n 3^n + 3(2^n) + \frac{7}{2} n^2 3^{n-2}$

Given $a_0 = 1, n = 0$

$1 = C_1 + 0 + 3 + 0$

$1 - 3 = C_1$

$\boxed{C_1 = -2}$

$a_1 = 4, n = 1$

$4 = C_1 3 + C_2 3 + 3(2) + \frac{7}{2} 3^{1-2}$

$4 = -6 + 3C_2 + 6 + 7/2(3)$

$3C_2 = 4 - 7/6$

$3C_2 = \frac{24-7}{6}$

$3C_2 = 17/6$

$\boxed{C_2 = 17/18}$

$\therefore a_n = -2(3^n) + \frac{17}{18} n(3^n) + 3(2^n) + \frac{7}{2} n^2 3^{n-2}$

$a_n = -2(3^n) + \frac{17}{2} n\, 3^{n-2} + 3(2^n) + \frac{7}{2} n^2 2^{n-2}$

Solve the r.r $a_{r+2} - 2a_{r+1} + a_r = 2^r . r^2$

Sol:

Given

$$a_{r+2} - 2a_{r+1} + a_r = 2^r . r^2 \quad \text{—— (1)}$$

The C.E is

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$x = 1, 1$

$\cancel{H.S = (C_1 + nC_2)1^n}$

$\cancel{= (C_1 + C_2 n)1^n}$

H.S: $(C_1 + rC_2)\alpha^r$

$(C_1 + rC_2)(1)^r$

$= (C_1 + rC_2$

~~Assume P.S as~~ $\cancel{a_r^{(P)} = A2^r}$

Assume P.S as $a_r^{(P)} = 2^r[b_0 + b_1 r + b_2 r^2]$

subs $a_r = 2^r[b_0 + b_1 r + b_2 r^2]$ in (1)

$2^{r+2}[b_0 + b_1(r+2) + b_2(r+2)^2] - 2[2^{r+1}(b_0 + b_1(r-1)$

$+ b_2(r+1)^2] + 2^r[b_0 + b_1 r + b_2 r^2] = 2^r . r^2.$

$2^r \cdot 2^2 [b_0 + b_1 r + 2b_1 + b_2 r^2 + 4b_2 + 4rb_2] - 2^r \cdot 2^2$
$[b_0 + b_1 r + b_1 + b_2 r^2 + b_2 + 2b_2 r] + 2^r [b_0 + b_1 r + b_2 r^2] = 2^r \cdot r^2$

$4[b_0 + b_1 r + 2b_1 + b_2 r^2 + 4b_2 + 4rb_2] - 4[b_0 + b_1 r + b_1 + b_2 r^2 + b_2 + 2b_2 r] + b_0 + b_1 r + b_2 r^2 = r^2.$

Equating the coeff of r^2

$$\cancel{4b_2} - \cancel{4b_2} + b_2 = 1$$
$$\boxed{b_2 = 1}$$

Equating the coeff of r'

$$\cancel{4b_1} + 16b_2 - \cancel{4b_1} - 8b_2 + b_1 = 0$$
$$16b_2 - 8b_2 + b_1 = 0$$
$$8b_2 + b_1 = 0$$
$$8 + b_1 = 0$$
$$\boxed{b_1 = -8}$$

Equating the coeffi of constal.

$$\cancel{4b_0} + 8b_1 + 16b_2 - \cancel{4b_0} - 4b_1 - 4b_2 + b_0 = 0$$
$$4b_1 + 12b_2 + b_0 = 0$$
$$4(-8) + 12(1) + b_0 = 0$$
$$-32 + 12 + b_0 = 0$$
$$-20 + b_0 = 0 \qquad \begin{array}{r}32\\12\end{array}$$
$$\boxed{b_0 = 20}$$

$\therefore a_r^{(P)} = 2^r [20 - 8r + r^2]$

$\therefore a_r = a_r^{(H)} + a_r^{(P)}$

$a_r = (C_1 + r C_2)(1)^r + 2^r [20 - 8r + r^2]$

GENERATING FUNCTION:-

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers then $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$= \sum_{n=0}^{\infty} a_n x^n$ is called generating function.

$1 - x + x^2 - x^3 \cdots = (1+x)^{-1} = \frac{1}{1+x}$

$1 + x + x^2 + x^3 \cdots (1-x)^{-1} = \frac{1}{1-x}$

General term	Generating function
$a_n = 1^n$	$\frac{1}{1-x}$
$a_n = (-1)^n$	$\frac{1}{1+x}$
$a_n = a^n$	$\frac{1}{1-ax}$
$a_n = (-a)^n$	$\frac{1}{1+ax}$
$a_n = n$	$\frac{x}{(1-x)^2}$
$a_n = n+1$	$\frac{1}{(1-x)^2}$
$a_n = (n+1)^2$	$\frac{x+1}{(1-x)^3}$
$a_n = n^2$	$\frac{x(x+1)}{(1-x)^3}$

Formula:-

1) $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$

2) $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$

3) $(1-ax)^{-1} = \sum_{n=0}^{\infty} a^n x^n$

4) $(1+ax)^{-1} = \sum_{n=0}^{\infty} (-a)^n x^n$

5) $(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$

6) $(1-x)^{-3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

Result: $G(x) = \sum_{n=0}^{\infty} a_n x^n$

1) $\sum_{n=0}^{\infty} a_{n+2} x^n = \frac{G(x) - a_0 - a_1 x}{x^2}$

2) $\sum_{n=0}^{\infty} a_{n+1} x^n = \frac{G(x) - a_0}{x}$

3) $\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 G(x)$

4) $\sum_{n=2}^{\infty} a_{n-1} x^n = x[G(x) - a_0]$

5) $\sum_{n=2}^{\infty} a_n x^n = G(x) - a_0 - a_1 x$

1. Solve the recurrence relation $a_{n+2} - 3a_{n+1} + 2a_n = 0$ by method of generating fn/: with initial condition $a_0 = 2$ and $a_1 = 3$.

Sol:

Given: $a_{n+2} - 3a_{n+1} + 2a_n = 0$ —— ①

multiply the equation ① by x^n and summing from $n=0$ to ∞.

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 3\sum_{n=0}^{\infty} a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2} x^{n+2} - \frac{3}{x}\sum_{n=0}^{\infty} a_{n+1} x^{n+1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\frac{1}{x^2}\left[a_2x^2+a_2x^3\cdots\right]-\frac{3}{x}\left[a_1x+a_2x^2\cdots\right]+2\sum_{n=0}^{\infty}a_nx^n=0$$

$$\frac{G(x)-a_0-a_1x}{x^2}-\frac{3\left[G(x)-a_0\right]}{x}+2G(x)=0$$

$$\frac{G(x)-a_0-a_1x-3x\left[G(x)-a_0\right]+2x^2G(x)}{x^2}=0$$

$$G(x)-a_0-a_1x-3xG(x)+3xa_0+2x^2G(x)=0$$

$$G(x)-2-3x-3xG(x)+3x(2)+2x^2G(x)=0$$

$$G(x)\left[1-3x+2x^2\right]-2-3x+6x=0$$

$$G(x)\left[2x^2-3x+1\right]=2-3x.$$

$$\therefore G(x)=\frac{2-3x}{2x^2-3x+1}$$

$$G(x)=\frac{2-3x}{(2x-1)(x-1)} \quad \text{——— (2)}$$

$$\frac{2-3x}{(2x-1)(x-1)}=\frac{A}{2x-1}+\frac{B}{x-1}$$

$$2-3x=A(x-1)+B(2x-1)$$

put $x=1$,

$$2-3(1)=0+B(2(1)-1)$$

$$2-3=B(2-1)$$

$$\boxed{-1=B}$$

put $x=1/2$,

$$2-3(1/2)=A(1/2-1)+0$$

$$2-3/2=A(-1/2)$$

$$1/2=-A/2$$

$$\boxed{A=-1}$$

$$\therefore \frac{2-3x}{(2x-1)(x-1)} = \frac{-1}{2x-1} + \frac{-1}{x-1}$$

$$\Rightarrow G(x) = \frac{-1}{2x-1} + \frac{-1}{x-1}$$

$$G(x) = \frac{1}{1-2x} + \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} a_n x^n = (1-2x)^{-1} + (1-x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 1^n x^n$$

$$a_n = 2^n + 1^n$$

2. Solve $a_{n+1} - a_n = 3^n$, $n \geq 0$, $a_0 = 1$ by G.F

Sol:

Given

$$a_{n+1} - a_n = 3^n \longrightarrow (1)$$

× by the equation (1) by x^n & summing 0 to ∞.

$$\sum_{n=0}^{\infty} a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3^n x^n$$

$$\frac{1}{x} \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3^n x^n$$

$$\left[\frac{a_1 x + a_2 x^2 \dots}{x}\right] - G(x) = \frac{1}{1-3x}$$

$$\left(\frac{G(x) - a_0}{x}\right) - G(x) = \frac{1}{1-3x}$$

$$\frac{G(x) - a_0 - x\,G(x)}{x} = \frac{1}{1-3x}$$

$$G(x) - 1 - x\,G(x) = \frac{x}{1-3x}$$

$$G(x)[1-x] - 1 = \frac{x}{1-3x}$$

$$G(x)[1-x] = \frac{x}{1-3x}+1$$

$$G(x)[1-x] = \frac{x+1-3x}{1-3x}$$

$$G(x)[1-x] = \frac{1-2x}{1-3x}$$

$$G(x) = \frac{1-2x}{(1-3x)(1-x)} \quad \text{——} \quad (2)$$

$$\frac{1-2x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$$1-2x = A(1-x) + B(1-3x)$$

put $x=1$,

$$1-2 = B(1-3)$$

$$-1 = B(-2)$$

$$\boxed{1/2 = B}$$

put $x = 1/3$.

$$1-2(1/3) = A(1-1/3) + B(1-3(1/3))$$

$$1-2/3 = A(1-1/3) + B(1-1)$$

$$3-2/3 = A(3-1/3)$$

$$1/3 = A(2/3)$$

$$1/3 \times 3/2 = A$$

$$\boxed{1/2 = A}$$

$$G(x) = \frac{1}{2}\left(\frac{1}{1-3x}\right) + \frac{1}{2}\left(\frac{1}{1-x}\right)$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\sum_{n=0}^{\infty} 3^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 1^n x^n$$

$$\therefore a_n = \frac{1}{2}(3)^n + \frac{1}{2}1^n$$

3) Solve the r.r for $a_n = 3a_{n-1}$ for $n = 1, 2, 3 \ldots\ldots,\ a_0 = 2.$

Sol:

Given:

$$a_n = 3a_{n-1}$$

$$a_n - 3a_{n-1} = 0 \quad \text{---} \ (1)$$

x^n in (1) and summing $n = 1$ to ∞.

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 0$$

$$[a_1 x + a_2 x^2 \ldots] - 3x[a_0 + a_1 x^1 \ldots] = 0$$

$$G(x) - a_0 - 3x\,G(x) = 0$$

$$G(x) - 2 - 3x\,G(x) = 0$$

$$G(x)[1-3x] - 2 = 0$$

$$G(x)[1-3x] = 2$$

$$G(x) = \frac{2}{1-3x}$$

$$\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 3^n x^n$$

$$\therefore a_n = 2(3^n)$$

Solve recurrence relation for $a_n - a_{n-1} = 2n,\ n \geq 1,$

$a_0 = 0$. by using G.F.

Sol:

Given:

$$a_n - a_{n-1} = 2n \quad \text{--- (1)}$$

multiply x^n in (1) and summing 1 to ∞.

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 2\sum_{n=1}^{\infty} n x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 2\sum_{n=1}^{\infty} n x^n$$

$$[a_1 x + a_2 x^2 \ldots] - x[a_0 + a_1 x + \ldots] = 2[x + 2x^2 + 3x^3 \ldots]$$

$$[G(x) - a_0] - x\,G(x) = 2x[1 + 2x + 3x^2 \ldots]$$

$$(G(x) - 0) - x\,G(x) = 2x(1-x^2)$$

$$G(x) - x\,G(x) = \frac{2x}{(1-x)^2}$$

$$G(x)[1-x] = \frac{2x}{(1-x)^2}$$

$$\therefore G(x) = \frac{2x}{(1-x)(1-x)^2}$$

$$G(x) = \frac{2x}{(1-x)^3} \qquad \because (1-x)^{-3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}$$

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$$

$$= \frac{2[x-1+1]}{(1-x)^3}$$

$$= \frac{2(x-1)}{(1-x)^3} + \frac{2}{(1-x)^3}$$

$$= \frac{-2(1-x)}{(1-x)^3} + \frac{2}{(1-x)^3}$$

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$$G(x) = \frac{-2}{(1-x)^2} + \frac{2}{(1-x)^3}$$

$$= -2(1-x)^{-2} + 2(1-x)^{-3}$$

$$\sum_{n=0}^{\infty} a_n x^n = -2\sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$$

$$\therefore a_n = -2(n+1) + (n+1)(n+2)$$

$$= (n+1)[-2+n+2]$$

$$a_n = n(n+1)$$

5) Use G.F to solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$.

Sol:

Given:

$$a_n = 4a_{n-1} - 4a_{n-2} + 4^n$$

$$a_n - 4a_{n-1} + 4a_{n-2} = 4^n \quad \text{--- (1)}$$

x^{ly} by x^n in (1) and summing $n=2$ to ∞.

$$\sum_{n=2}^{\infty} a_n x^n - 4\sum_{n=2}^{\infty} a_{n-1}x^n + 4\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2}^{\infty} 4^n x^n$$

$$\sum_{n=2}^{\infty} a_n x^n - 4x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = \sum_{n=2}^{\infty} 4^n x^n$$

$$[a_2x^2 + a_3x^3 \ldots] - 4x[a_1x + a_2x^2 \ldots] + 4x^2[a_0 + a_1x \ldots]$$

$$= (4x)^2 + (4x)^3 + (4x)^4 \ldots$$

$$[G(x) - a_0 - a_1x] - 4x[G(x) - a_0] + 4x^2G(x) = (4x)^2[1 + 4x + (4x)^2 \ldots]$$

$$[G(x) - 2 - 8x] - 4x[G(x) - 2] + 4x^2G(x) = 16x^2[1-4x]^{-1}$$

$$G(x) - 2 - \cancel{8x} - 4x\,G(x) + \cancel{8x} + 4x^2 G(x) = \frac{16x^2}{(1-4x)}$$

$$G(x)\left[1-4x+4x^2\right] - 2 = \frac{16x^2}{(1-4x)}$$

$$G(x)\left[1-4x+4x^2\right] = \frac{16x^2}{(1-4x)} + 2$$

$$G(x)\left[(2x-1)(2x-1)\right] = \frac{16x^2 + 2(1-4x)}{(1-4x)}$$

$(2x-1)(2x-1)$

$4x^2-2x-2x+1$

$4x^2-4x+1$

$$G(x)\left[(1-2x)(1-2x)\right] = \frac{16x^2+2-8x}{1-4x}$$

$$\therefore G(x) = \frac{16x^2-8x+2}{(1-4x)(1-2x)^2}$$

$$\frac{16x^2-8x+2}{(1-4x)(1-2x)^2} = \frac{A}{1-4x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

$$\frac{16x^2-8x+2}{(1-4x)(1-2x)^2} = \frac{A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x)}{(1-4x)(1-2x)^2}$$

$$16x^2-8x+2 = A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x)$$

put $x = 1/2$

$$16(1/2)^2 - 8(1/2) + 2 = 0 + 0 + C(1-4(1/2))$$

$$\frac{16}{4} - 4 + 2 = C(1-2)$$

$$4-4+2 = C(-1)$$

$$\boxed{C = -2}$$

Put $x = 1/4$

$$16(1/4)^2 - 8(1/4) + 2 = A(1-2(1/4))^2 + 0 + 0$$

$$16/16 - 2 + 2 = A(1-1/2)^2$$

$$1 - 2 + 2 = A(1/2)^2$$

$$1 = A(1/4)$$

$\boxed{A=4}$

Equating the coeff of x^2

$x=0$,

~~$2 = A(1) + B(1) + C$~~

$$0-8(0)+2 = A(1-0)^2 + B(1-0)(1-0) + C(1-0)$$

$$2 = A+B+C$$

$$2 = 4+B-2$$

$$2+2 = 4+B$$

$$4 = 4+B$$

$$4-4 = B$$

$$\boxed{B=0}$$

sub A, B, C in ①

$$\frac{16x^2-8x+2}{(1-4x)(1-2x)^2} = \frac{4}{1-4x} + 0 + \frac{-2}{(1-2x)^2}$$

$$\therefore G(x) = \frac{4}{1-4x} - 2\frac{1}{(1-2x)^2}$$

$$\sum_{n=0}^{\infty} a_n x^n = 4\sum_{n=0}^{\infty} 4^n x^n - 2(1-2x)^{-2}$$

$$= 4\sum_{n=0}^{\infty} 4^n x^n - 2\sum_{n=0}^{\infty}(n+1)(2x)^n$$

$(1-2x)^{-2} = 1+2(2x)+3(2x)^2 \ldots$

$= \sum_{n=0}^{\infty}(n+1)(2x)^n$

$$= 4\sum_{n=0}^{\infty} 4^n x^n - 2\sum_{n=0}^{\infty}(n+1)2^n x^n$$

$$a_n = 4(4^n) - 2(2^n)(n+1)$$