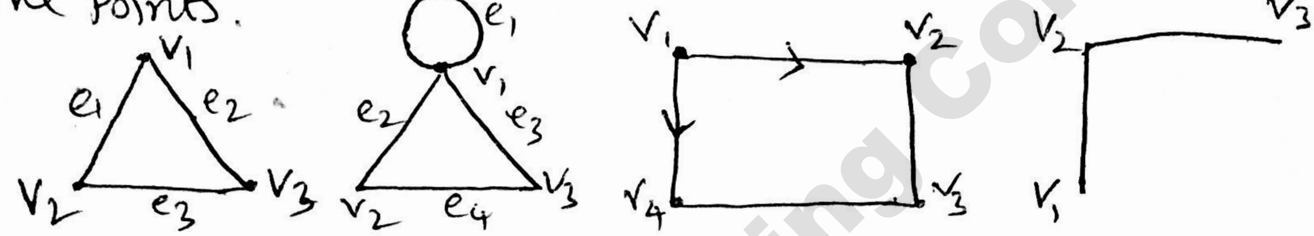
UNIT-3 GRAPHS

Defn :- A graph is a pair of sets (V, E) where V is a non-empty set. The set V is called the set of vertices or nodes and the set E is called the set of Edges.

Each vertex is represented by a point and each edge is represented by an arc or st. line joining the points.

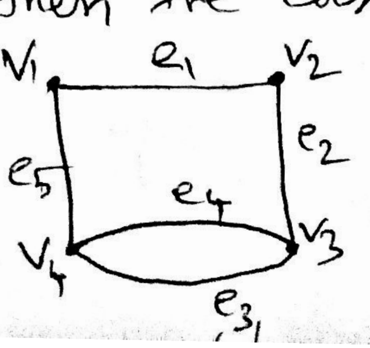


Undirected graph :- Let $G = (V, E)$ be a graph. If the elements of E are unordered pairs of vertices of G then G is called an undirected graph.

Directed graph (or) Digraph :- Let $G = (V, E)$ be a graph. If the elements of E are ordered pairs of vertices, then the graph G is called a digraph.

Self Loop :- If there is an edge from v_i to v_i then that edge is called self loop or simply loop.

Parallel Edges :- If two edges have same end points then the edges are called parallel edges.



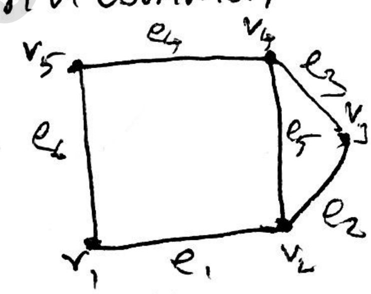
The edge e_3 and e_4 are called parallel edges since e_3 and e_4 have the same pair of vertices (v_3, v_4) as their terminal vertices.

Multigraph:- A graph which has more them one edge between a pair of vertices is called a multigraph.

Simple graph:- A graph with no self loops and no parraller edges is called a simple graph.

Incident:- If the vertex v_i is an end vertex of some edge e_k then e_k is said to be incident with v_i

Adjacent edges and vertices:- Two edges are said to be adjacent if they are incident on a common vertex. Ex:- e_4 and e_6 are adjacent.



Two vertices v_i & v_j are said to adjacent if $v_i v_j$ is an edge of the graph. Ex: v_1 & v_5 are adjacent vertices

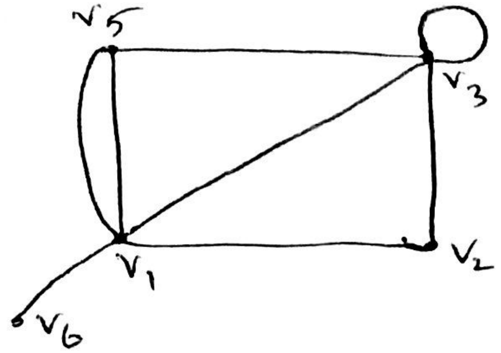
Indegree:- In a diagraph G, the no. of edges ending at vertex v of G is called the indegree of v. Indegree of v denoted by $d^{(-)}(v)$.

Out degree:- let G be a diagraph and v be vertex of G. The out degree of v is the no. of edges begining at v and is denoted by $d^{(+)}(v)$.

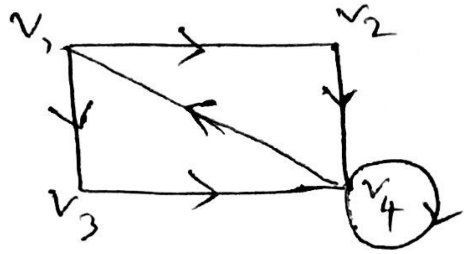
Degree of a vertex:- In an undirected graph G, the degree of a vertex v is defined as the no. of edges incident with v with self loops counted twice and is denoted by $d(v)$.

In case of a diagraph G, $d(v) = d^{(-)}(v) + d^{(+)}(v)$.

Ex:-



$d(v_1) = 5$	$d(v_4) = 0$
$d(v_2) = 2$	$d(v_5) = 3$
$d(v_3) = 5$	$d(v_6) = 1$



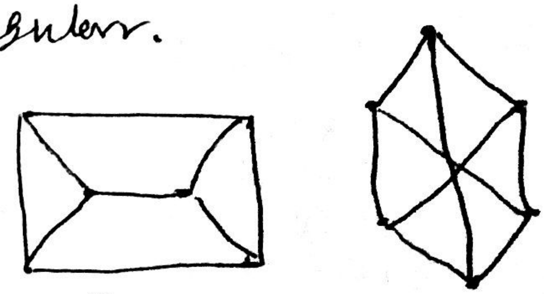
Vertex	Indegree	outdegree
v_1	1	2
v_2	1	1
v_3	1	1
v_4	4	1

Isolated vertex :- A vertex of degree 0 in a graph (or) A vertex having no edges incident on it is called isolated.

Pendent vertex: A vertex of a graph with degree one is called a pendent vertex.

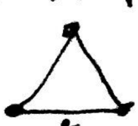
Regular graph :- If every vertex of a simple graph has the same degree, then the graph is called a regular. If every vertex in a regular graph has degree k, then the graph is called k-regular.

2-regular graph

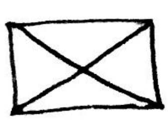


3-regular graph.

Complete graph :- A simple graph G in which every pair of vertices are connected by an edge is called a complete graph. If G is a complete graph of n vertices then it is denoted by k_n.



k_3



k_4

Theorem 1 (The Handshaking theorem)

(i) The sum of degrees of the vertices of an un-directed graph G is twice the no. of edges in G.

i.e, $\sum_{i=1}^{n} deg(V_i) = 2|E|$ and

(ii) If G is a directed graph, $\sum_{i=1}^{n} d^{(-)}(v_i) = \sum_{i=1}^{n} d^{(+)}(v_i)$ and

$\sum d^{(-)}(v_i) + \sum d^{(+)}(v_i) = 2|E| = 2$ (no. of edges in G)

Proof: Let G be an undirected graph. Each edge of G is incident with 2 vertices and hence contributes 2 to the sum of degrees of all the vertices of the undirected graph G.

∴ The sum of degrees of all the vertices in G is twice the no. of edges in G.

i.e, $\sum_{i=1}^{n} d(V_i) = 2|E| = 2$ (no. of. edges in G).

(ii) Let G be digraph and e be an edge associated with a vertex pair (V_i, V_j). The edge e contributes one to the out degree of V_i and one to the in-degree of V_j. This is true for all the edges in G.

Hence $\sum_{i=1}^{n} d^{(-)}(v_i) = \sum_{i=1}^{n} d^{(+)}(V_i) = |E|$

$\therefore \sum d(V) = 2|E|$.

Theorem 2:- In an undirected graph G, the no. of vertices of odd degree is even.

Proof:- Let $G = (V, E)$ be an undirected graph

The vertex set V is divided into 2 sets W and U.

where W: set of odd degree vertices and

U: set of even degree vertices.

Then $\sum_{v_i \in V} deg(v_i) = \sum_{v_i \in W} deg(v_i) + \sum_{v_i \in U} deg(v_i)$

$$\sum_{v_i \in W} deg(v_i) = \sum_{v_i \in V} deg(v_i) - \sum_{v_i \in U} deg(v_i)$$

Since $\sum_{v_i \in V} deg(v_i)$ is even and $\sum_{v_i \in U} deg(v_i)$ is also even

$\therefore \sum_{v_i \in W} deg(v_i)$ is = even number

$\therefore$ Eeach $deg(v_i)$ on LHS is odd and the no. of summands must be even.

$\therefore$ The no. of odd degree vertices in G is even.

Theorem 3:- The maximum no. of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$

Soln: We prove this theorem by the Principle of Mathematical Induction. For $n=1$, a graph with one vertex has no edge.

$\therefore$ The result is true for $n=1$.

For $n=2$, a graph with 2 vertices may have atmost one edge.

$\therefore \frac{2(2-1)}{2} = 1$. The result is true for $n=2$.

Assume that the result is true for $n=k$. ie, a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges.

To prove for $n=k+1$,

~~when $n=k+1$, let G be a graph having 'n' vertices and~~

Now, we construct a simple graph from the graph having k vertices with the maximum edges $\frac{k(k-1)}{2}$, by introducting a new $(k+1)^{th}$ vertex v_{k+1} we add k edges to the graph by connecting the new vertex v_{k+1} and each of the remaining k vertices.

$\therefore$ The max. possible no. of edges $= \frac{k(k-1)}{2} + k$

$$= \frac{k^2-k+2k}{2} = \frac{k^2+k}{2}$$

$\therefore$ The result is true for $n=k+1$

$$= \frac{k(k+1)}{2} = \frac{(k+1)(k+1-1)}{2}$$

Walk :

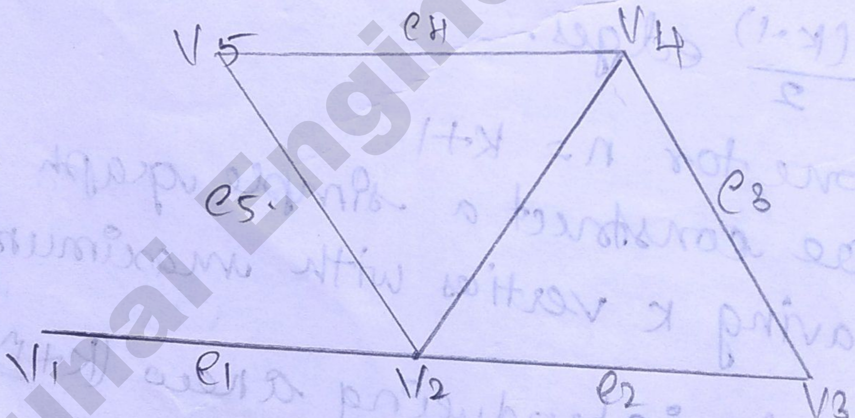
A walk in a graph is defined as a finite alternating sequences of vertices and edges.

i.e, $V_0 e_1 V_1 e_2 V_2 \dots e_n V_n$.

Begining and ending with vertices. If V_0 not equal to V_n then the walk is open walk.

And If $V_0 = V_n$ then the walk is closed walk.

Ex:

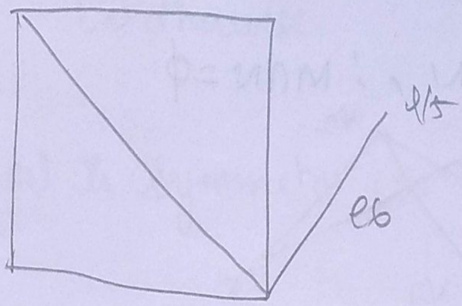


Path:-

A open walk with no vertex appear more than one is called a path. (Simple path or Elementary path).

The no. of edges in a path is called the length of a path.

Ex:



closed path:

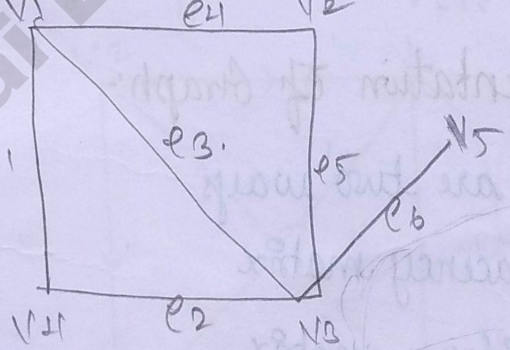
A path between a vertex and itself is called closed path.

Circuit:

A closed path in which all the edges are distinct is a circuit.

Cycle:

A circuit in which all the vertices are distinct is called a cycle.



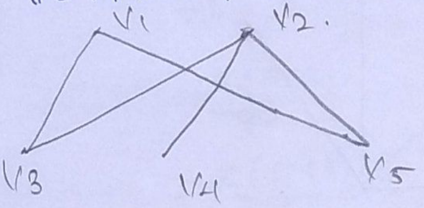
Circuit: $V_1 e_3 V_3 e_5 V_2 e_4 V_1$.

Bipartite Graph:

A Bipartite graph is an undirected graph whose set of vertices can be partitioned into two sets. M, N. is such a way that each edge joins a vertex in M to a vertex in N.

And no edge joins either to vertices in M or to vertices in N.

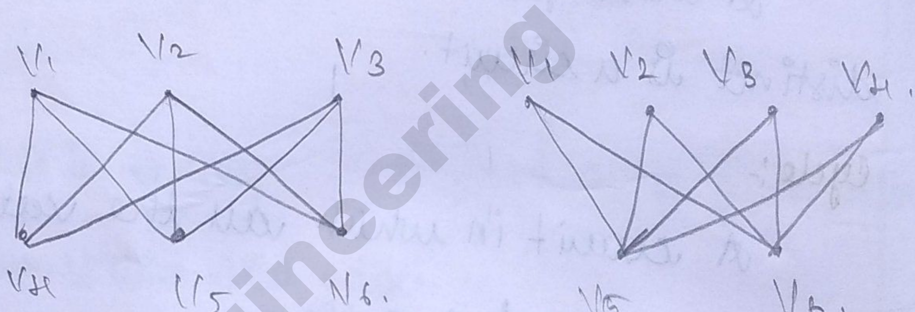
Ex:
Let, $V = M \cup N$, ; $M \cap N = \phi$



Complete bipartite graph:-

A complete bipartite graph is a bipartite graph in every vertex of M is adjacent to every vertex of N. It is denoted by $K(M, N)$

Ex:



$K_{3,3}$ $K_{4,2}$

Matrix representation of Graph:-

There are two ways
1. Adjacency matrix
2. Incidence matrix.

Adjacency matrix:-

Let G be a graph with n vertices and no parallel edges. The adjacency matrix of G is defined by $A(G) = (a_{ij})$

where, $a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ \& } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$

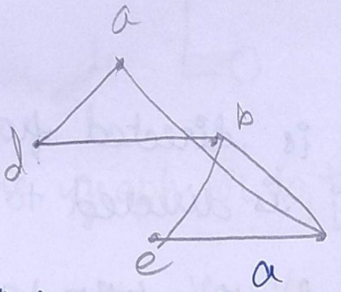
For directed graph,

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

Note :-

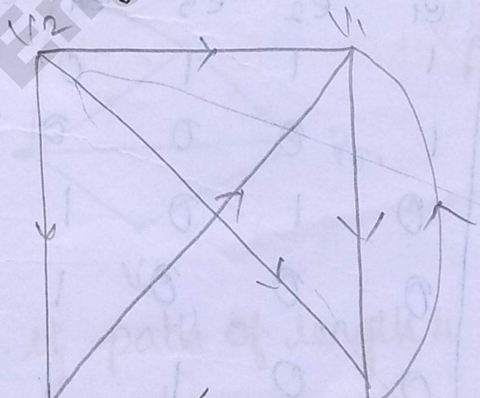
Here, $A(G)$ is Symmetry

Ex:



$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Find the adjacency matrix of following graph:-



$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4. \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix:

Let G be a Graph with N vertices and M edges. The incidence matrix defined by $B(G) = (b_{ij})$

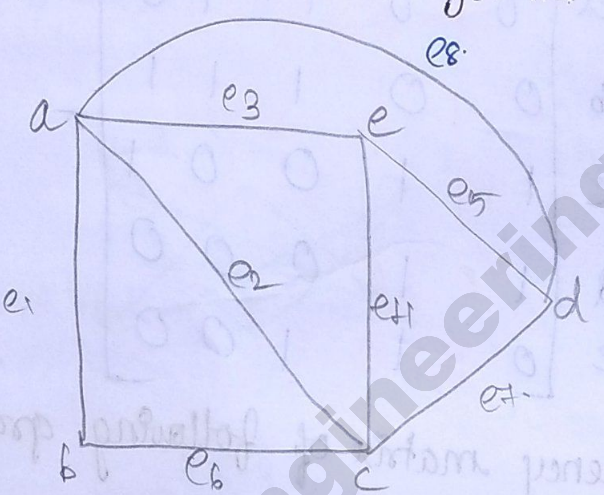
where

$$b_{ij} = \begin{cases} 1, \text{ if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex} \\ 0, \text{ otherwise.} \end{cases}$$

For directed graph,

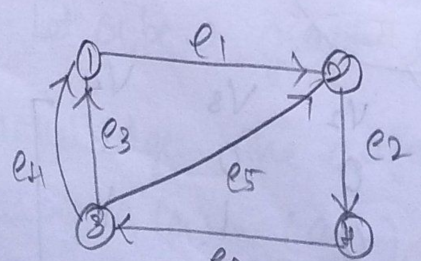
$$b_{ij} = \begin{cases} 1, \text{ if edge } e_j \text{ is directed from } V_i \\ -1 \text{ if edge } e_j \text{ is directed to } V_i. \end{cases}$$

Write the incidence matrix for the following graph:-



$$B(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(2).

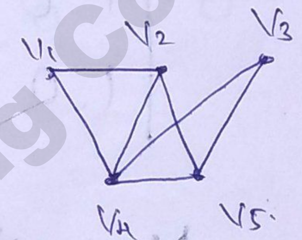


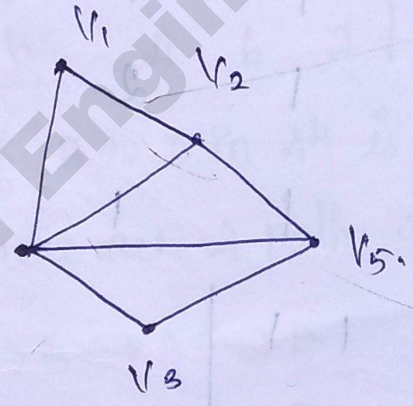
Find the incident matrix of the following graph:-

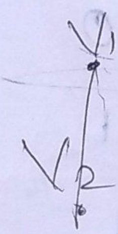
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6. \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

1. Draw the graph G to the following adjacency matrix.

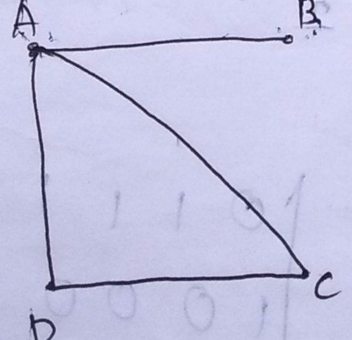
$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}\begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$







2. Find the no. of path of length 4 from B to D in the following graph.



Sol:

The adjacency matrix is

$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = A \cdot A.$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} (1+1+1) & (0) & (1) & (1) \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{vmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & \cancel{3}\,4 & 4 \\ 3 & 0 & 1 & 1 \\ 4 & 1 & 2 & 3 \\ 4 & 1 & 3 & 2 \end{vmatrix}$$

$A^4 = A^3 \cdot A$

$$= \begin{vmatrix} 2 & 3 & 4 & 4 \\ 3 & 0 & 1 & 1 \\ 4 & 1 & 2 & 3 \\ 4 & 1 & 3 & 2 \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$A^4 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{vmatrix} \cancel{10}\,11 & 2 & 6 & 6 \\ 2 & 3 & 4 & \boxed{4} \\ 6 & 4 & 7 & 6 \\ 6 & 4 & 6 & 7 \end{vmatrix} \end{array}$$

The entry of B to D in A^4 is 4.

Hence there are 4 path of length 4 from B to D.

i) $B \rightarrow A \rightarrow B \rightarrow A \rightarrow D$.

ii) $B \rightarrow A \rightarrow D \rightarrow C \rightarrow D$

iii) $B \rightarrow A \rightarrow D \rightarrow A \rightarrow D$

iv) $B \rightarrow A \rightarrow C \rightarrow A \rightarrow D$.

Theorem 4 :

If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.

Proof :

Let $2n$ be the number of vertices of the given graph.

Let n_e be the number of edges of the given graph. ...(1)

By Handshaking theorem, we have

$$\sum_{i=1}^{2n} \deg V_i = 2n_e$$

$$\Rightarrow \quad 2nk = 2n_e \qquad \text{(Using (1))}$$

$$\Rightarrow \quad n_e = nk$$

$\Rightarrow$ number of edges = multiple of k.

$\therefore$ The number of edges of the given graph is a multiple of k.

Example 1 : ***How many edges are there in a graph with ten vertices each of degree six.***

Solution :

Let e be the number of edges of the graph.

$$\begin{aligned} 2e &= \text{Sum of all degrees} \\ &= 10 \times 6 \\ &= 60 \\ 2e &= 60 \\ e &= 30 \end{aligned}$$

$\therefore$ There are 30 edges.

Example 2 : ***Can a simple graph exist with 15 vertices each of degree 5.***

Solution :

$$2e = \sum d(v)$$

$$2e = 15 \times 5$$

$$= 75$$

$$\therefore \quad e = \frac{75}{2}$$

Which is not an integer.

$\therefore$ Such a graph does not exist.

(or)

By a theorem (Theorem 2), in a graph the number of odd degree vertices is even. Therefore, it is not possible to have 15 vertices, which is of odd degree.

$\therefore$ Such a graph does not exist.

Example 3 : ***For the following degree sequences, 4, 4, 4, 3, 2 find if there exist a graph or not.***

Solution :

$$\left.\begin{array}{l}\text{Sum of the degree of}\\ \text{all vertices}\end{array}\right\} = 4 + 4 + 4 + 3 + 2$$

$$= 17$$

Which is an odd number.

$\therefore$ Such a graph does not exist.

Example 4 : ***How many vertices does a regular graph of degree 4 with 10 edges have.***

Solution :

$$\sum d(v) = 2e$$

Let 'n' be the number of vertices and 'e' is the number of edges.

$$\therefore \quad 4n = 2 \times 10 \qquad \left(\begin{array}{l}\therefore \text{ In a regular}\\ \text{graph each vertices}\\ \text{is of same degree}\end{array}\right)$$

$$\Rightarrow \quad n = 5$$

$\therefore$ There are 5 vertices in a regular graph of degree 4 with 10 edges.

Example 5 : ***Does there exist a simple graph with five vertices of the following degrees? If so draw such graph***

(a) 1, 1, 1, 1, 1 ***(b) 3, 3, 3, 3, 2***

Solution :

(*a*) We know that in any graph the number of odd degree vertices is always even.

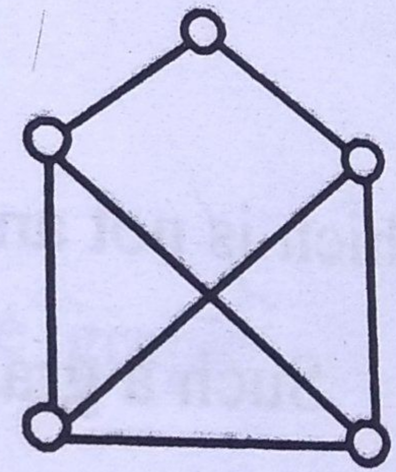
In case (*a*) number of odd degree vertices is 5 (not an even)

$\therefore$ Such a graph does not exist.

(*b*) For case (*b*),

$$\text{Sum of degree} = 14 = \text{even}$$

$\therefore$ The graph exist. The graph is



Isomorphism:-

If two graphs G and G' are isomorphic if there is consistent $f : V(G) \to V(G')$

Then,

i) f is ~~1 to 1~~ one ~~to~~ - one.

ii) f is on ~~two~~. to.

iii) f presence adjacency

(or)

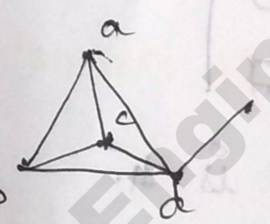
If graphs are isomorphic. if

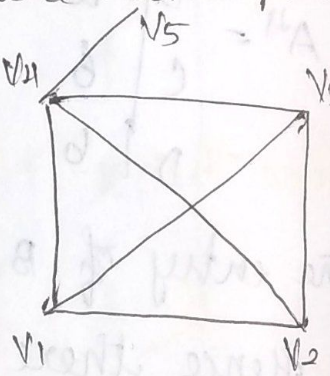
i) They have same number of vertices.

ii) They have " " " edges.

iii) Equal no. of vertices with same degree.

Show that the following graph are isomorphic.

$G =$ 

$G' =$ 

Sol:

Both graphs are 5 vertices & 7 edges.

In the graph G.

$d(a) = 3$, $d(b) = 3$, $d(c) = 3$, $d(d) = 4$

$d(e) = 1$

In the Graph G',

$d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 3$, $d(V_4) = 4$

$d(V_5) = 1$.

$\therefore$ The adjacency matrixes are ~~A(G)~~

$$A(G) = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 1 & 0 \\ c & 1 & 1 & 0 & 1 & 0 \\ d & 1 & 1 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 0 \end{array}$$

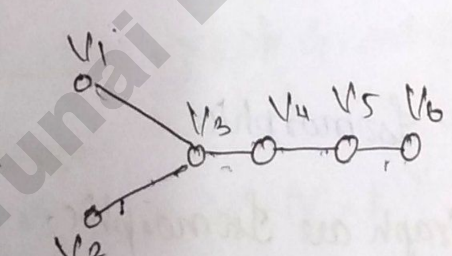
$$A(G') = \begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & 0 & 1 & 1 & 1 & 0 \\ V_2 & 1 & 0 & 1 & 1 & 0 \\ V_3 & 1 & 1 & 0 & 1 & 0 \\ V_4 & 1 & 1 & 1 & 0 & 1 \\ V_5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$\therefore$ The adjacency matrices are same.

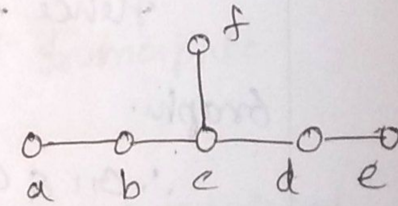
Hence the given two Graphs are Isomorphic.

Show that the Graph G_1, G_2 are not isomorphic.

$G_1 =$



$G_2 =$



Sol:

Both graphs are 6 vertices and 5 edges.

In the Graph G_1

$d(V_1) = 1$, $d(V_2) = 1$, $d(V_3) = 3$, $d(V_4) = 2$

$d(V_5) = 2$, $d(V_6) = 1$.

In the Graph G_2,

$d(a) = 1$, $d(b) = 2$, $d(c) = 3$, $d(d) = 2$, $d(e) = 1$

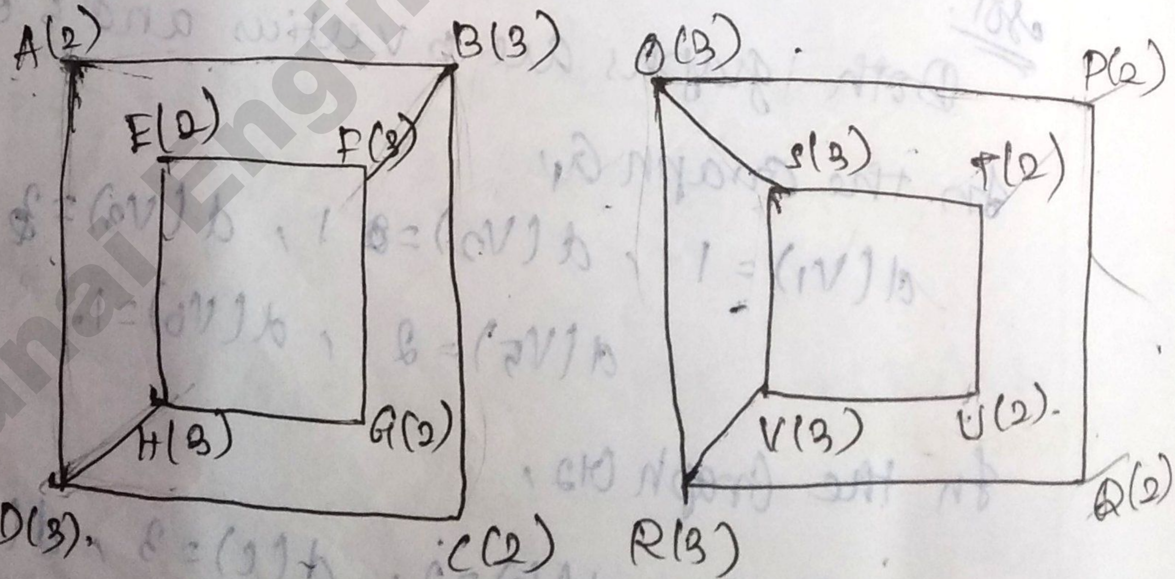
$d(f) = 1$.

In the Graph G_1 vertex V_3 of degree 3 and there are two pendent vertices adjacent to V_3 in G_1. But in the graph G_2 the vertex c is of degree 3 and it has only one pendent vertex adjacent to it.

Hence adjacency is not preserved in the Graphs.

$\therefore G_1$ & G_2 are not Isomorphic.

2. Determine whether the Graph are Isomorphic.



Both the Graphs are 8 vertices & 10 edges.

In the Graph 1,

$d(A)=2$; $d(B)=3$, $d(c)=2$, $d(d)\ 3$, $d(e)=2$

$d(f)=3$, $d(G)=2$, $d(h)=3$.

In the Graph 2,

$d(O)=3$, $d(P)=2$, $d(Q)=2$, $d(R)=3$,

$d(S)=3$, $d(T)=2$, $d(U)=2$, $d(V)=3$.

In the Graph G_1 vertex A of degree 2 must correspond to P, Q, T, U in G_2. which are of degree 2.

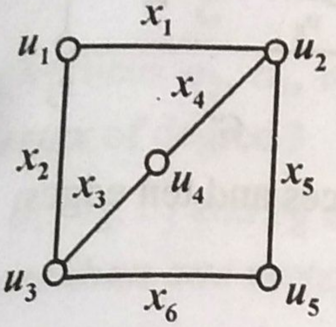
Also, the vertex P, Q, T, U are adjacent to another vertex of degree 2. in G_2.

But, A is not adjacent to any vertex of degree 2.

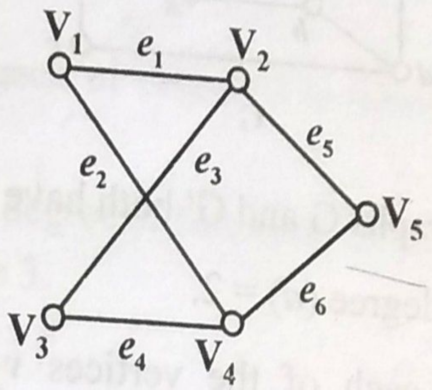
Hence there is no correspondence vertex A.

∴ The two graph G_1 and G_2 are not Isomorphic.

Example 2 : ***Check the given 2 graphs G and G′ are Isomorphic or not.***



G



G′

Solution :

The number of vertices **(5)** and number of edges **(6)** are same.

The degree sequence are same. Since, in G we have the vertices u_2 and u_3 of degree 3. They must be mapped to the vertices v_2 and v_4 in G′.

Define a mapping:

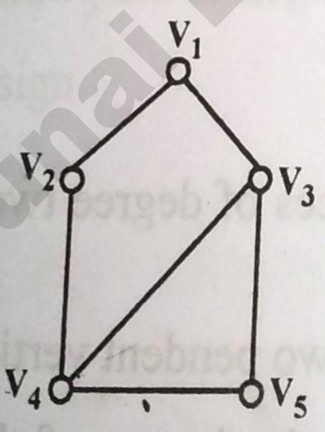
$$u_1 \to v_1,\ u_3 \to v_2,\ u_5 \to v_3,\ u_2 \to v_4 \text{ and } u_4 \to v_5$$

Then the edges. x_2, x_1, x_6, x_5, x_3,and x_4 are mapped into e_1, e_2, e_3, e_4, e_5 and e_6.

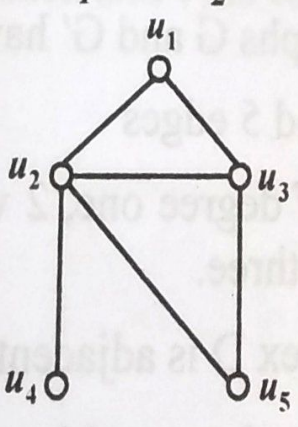
Therefore, there is a 1 – 1 correspondence between the vertices and edges.

Therefore, the given 2 graphs G and G′ are Isomorphic.

Example 3 : ***Check the 2 given graphs G_1 and G_2 are Isomorphic or not.***



G



G′

Solution :

The 2 graphs G and G′ have same number of vertices **(5)** and same number of edges **(6)**.

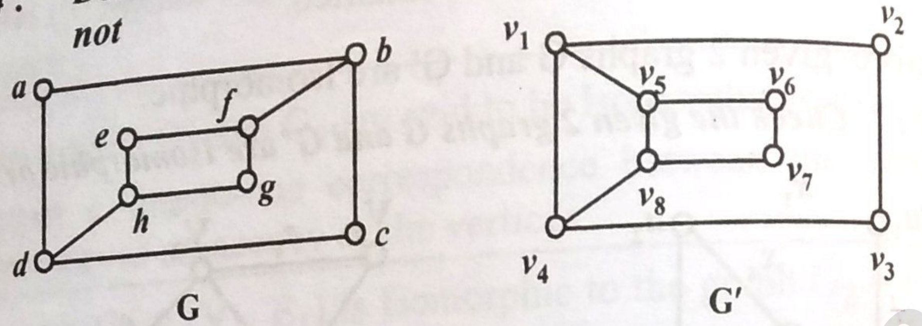
But, there is no one-to-one correspondence between edges in G and G′.

For,

The graph G have the degree sequence 2, 2, 2, 3, 3. But the degree sequence of G′ is 1, 2, 2, 3, 4.

Therefore, G and G′ are not Isomorphic.

Example 4 : ***Determine whether the graphs given below are Isomorphic or not***



Solution :

The graphs G and G′ both have eight vertices and ten edges.

In G, degree $(a) = 2$.

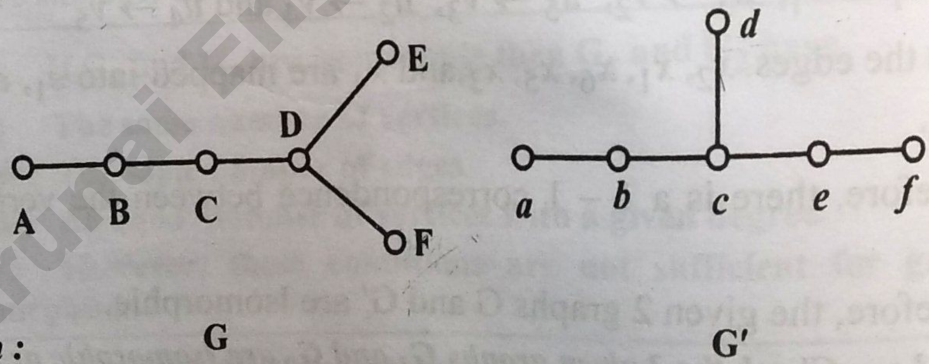
Since each of the vertices v_2, v_3, v_6 and v_7 is of deg 2 in G′.

Therefore, a in G must correspond to either v_2, v_3, v_6 and v_7 of G′.

Each of the vertices v_2, v_3, v_6 and v_7 in G′ are adjacent to another vertex of degree 2 in G′, which is not true for a in G.

Therefore G and G′ are not Isomorphic.

Example 5 : ***Chcek the given two graphs G and G′ are Isomorphic or not.***



Solution :

Here both the graphs G and G′ have

(1) 6 vertices and 5 edges

(2) 3 vertices of degree one, 2 vertices of degree two and 1 vertices D of degree three.

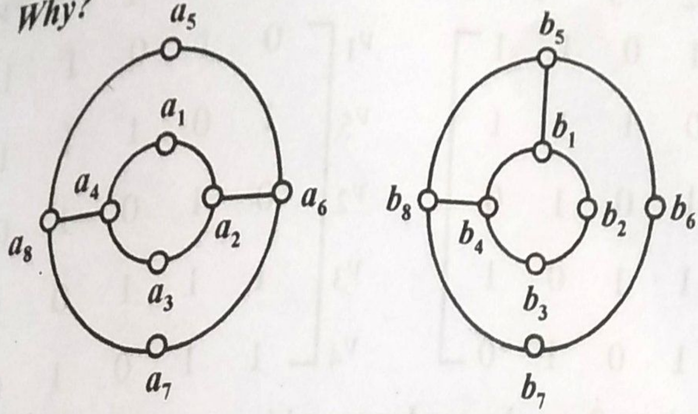
But in G, the vertex D is adjacent to two pendent vertices (E and F).

If G and G′ were Isomorphic, then the image of this vertex in G′ should be adjacent of two pendent vertices in H.

But in G′, there is no vertex which is adjacent to two pendent vertices.

Hence G and H are not Isomorphic.

Example 6 : ***Are the two graphs given in the following figure Isomorphic? Why?***



G G′

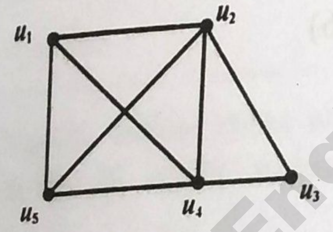
Solution :

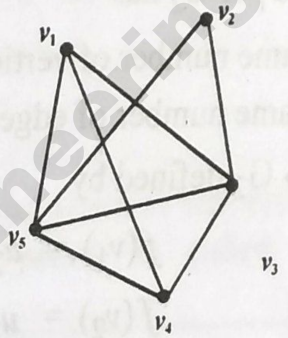
In G, the vertices a_2, a_4, a_6 and a_8 each of degree 3 is adjacent to ***exactly one vertex*** of degree 3.

But in G′ b_1, b_4, b_5 and b_8 are each of degree 3. But these vertices are adjacent to ***more than one vertex*** of degree 3.

∴ G and G′ are not Isomorphic.

Example 7 : ***Determine whether the following pairs of graphs are isomorophic.***





Solution :

The given 2 graphs have

(1) Same number of vertices (5)

(2) Same number of edges (8)

Moreover, in the given diagram u_1 and u_5 are of degree 3 each, u_2 and u_4 are of degree 4 each and u_3 is degree 2. Similarly v_1 and v_4 are of degree 3 each, v_3 and v_5 are of degree 4 each and v_2 is of degree 2.

Now, if we assign

$u_1 \to v_1$

$u_2 \to v_5$

$u_3 \to v_2$

$u_4 \to v_3$

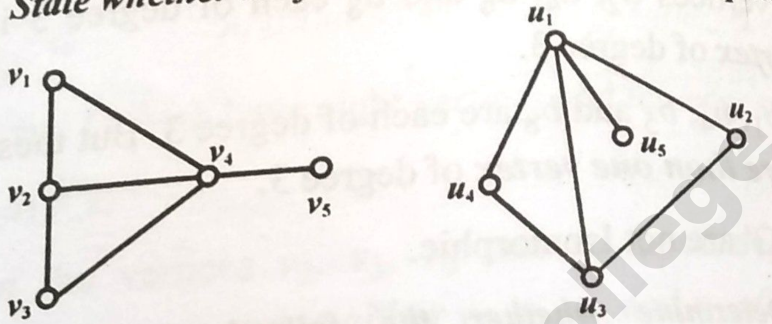
$u_5 \to v_4$

then the adjacency is preserved, which is evidently given by their adjacency matrix.

$$\begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 0 & 1 & 1 \\ u_2 & 1 & 0 & 1 & 1 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 \\ u_4 & 1 & 1 & 1 & 0 & 1 \\ u_5 & 1 & 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{c|ccccc} & v_1 & v_5 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 & 1 \\ v_5 & 1 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 0 & 1 & 0 \end{array}$$

$\therefore$ The given 2 graphs are Isomorphic.

Example 8 : ***State whether the following graphs are isomorphic or not.***



Solution :

Here both G_1 & G_2 has

(1) Same number of vertices (5)

(2) Same number of edges (6)

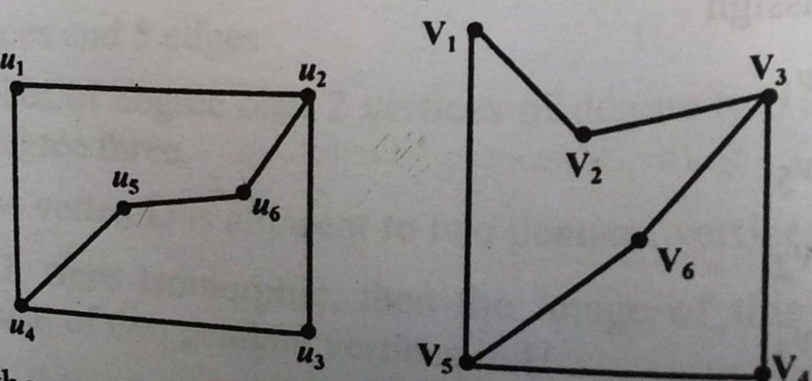
If $f : G_1 \to G_2$ defined by

$$f(v_1) = u_2$$
$$f(v_2) = u_3$$
$$f(v_3) = u_4$$
$$f(v_4) = u_1$$
$$f(v_5) = u_5$$

then f is bijective and also preserves adjacency.

$\therefore$ The given 2 graphs are Isomorphic.

Example 9 : ***Determine whether the graphs G and H are isomorphic.***



Solution :

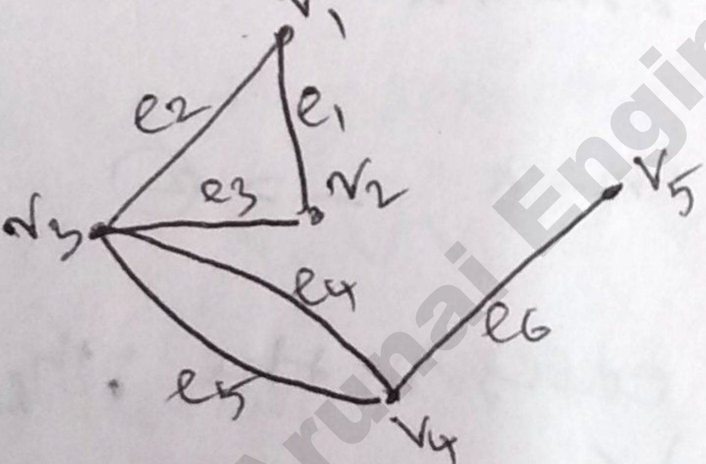
Both the graph G & H have

(1) Same number of vertices (6)

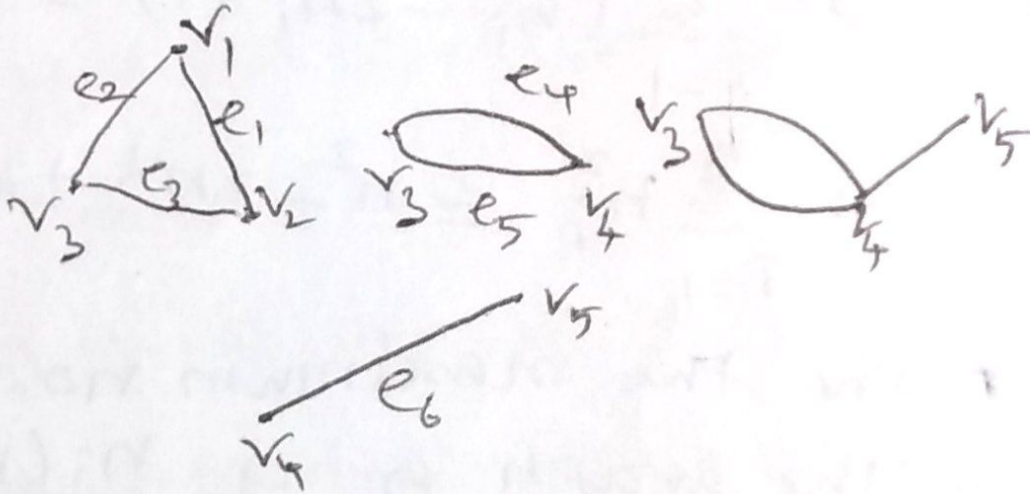
(2) Same number of edges (7)

A Subgraph:- A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$

In otherwords a graph H is said to be a subgraph of G if all the vertices and all the edges of H are in G and if the adjacency is preserved in H exactly as in G.



The subgraph are

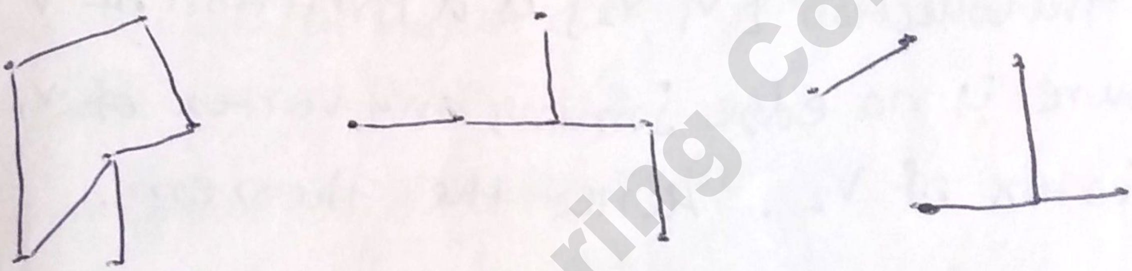


Note:- (i) every graph is own subgraph
(ii) A single vertex is subgraph

Connected Graph:- A graph G is said to be connected if every pair of vertices in G are joined by a path. If G is not connected then G is called a disconnected graph.

A maximal connected subgraph of G is called a components of G. If G is disconnected then G has atleast two components. clearly a graph G is connected if it has exactly one component.

Ex:-



Connected graph Disconnected graph.

Theorem-1 A graph G is connected iff for any partition of V into subsets V_1 and V_2 there is an edge joining of V_1 to a vertex of V_2.

Proof: Let G be a connected graph and $V = V_1 \cup V_2$ be a partition of V into two subsets. Let $u \in V_1$ and $v \in V_2$. Since the graph G is connected a path in G say. $u = v_0, v_1, v_2 \dots u_n = v$

Let i be the least positive integer such that $v_i \in V_2$. Then $v_{i-1} \in V_1$ and the vertices v_{i-1}, v_i are adjacent.

Thus there is an edge joining $v_{i-1} \in V_1$ and $v_i \in V_2$

Conversely :- Let G be a disconnected graph. then G contains atleast 2 components.

Let V_1 be the set of all vertices of one components and V_2 be the set of remaining vertices of G.

Clearly $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \Phi$

$\therefore$ the collection $\{V_1, V_2\}$ is a partition of V and there is no edge joining any vertex of V_1 to any vertex of V_2. Hence the theorem.

Theorem 2 :- A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof :- Let $n_1, n_2 \ldots n_k$ be the no. of vertices in each of the k components of a graph G.

Then $n_1 + n_2 + \ldots + n_k = n$, where $n_i \geq 1$

$$\sum_{i=1}^{k} n_i = n \rightarrow (1)$$

Now $\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = \sum_{i=1}^{k} n_i - k$

ie, $$\sum_{i=1}^{k} n_i - 1 = n - k$$

squaring on bothsides

$$\left[\sum_{i=1}^{k} n_i - 1 \right]^2 = (n-k)^2 = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i=ij} (n_i - 1)(n_j - 1) = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 \le n^2 - 2nk + k^2 \quad [\because n_i \ge 1]$$

$$\Rightarrow \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \le n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2 - 2nk + k^2 + 2n - k \quad \rightarrow (2)$$

Now the maximum no. of edges in the i^{th} component of the graph G is $n_i(n_i - 1)/2$

$\therefore$ The maximum no. of edges of G

$$= \frac{1}{2} \sum_{i=1}^{k} n_i(n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n \quad \text{by (1)}$$

$$\le \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2} n \quad \because \text{by (2)}$$

$$\le \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k - n)$$

$$\le \frac{1}{2}(n^2 - 2nk + k^2 + n - k)$$

$$\le \frac{1}{2}[(n-k)^2 + n - k]$$

$$\le \frac{1}{2}[(n-k)(n-k+1)]$$

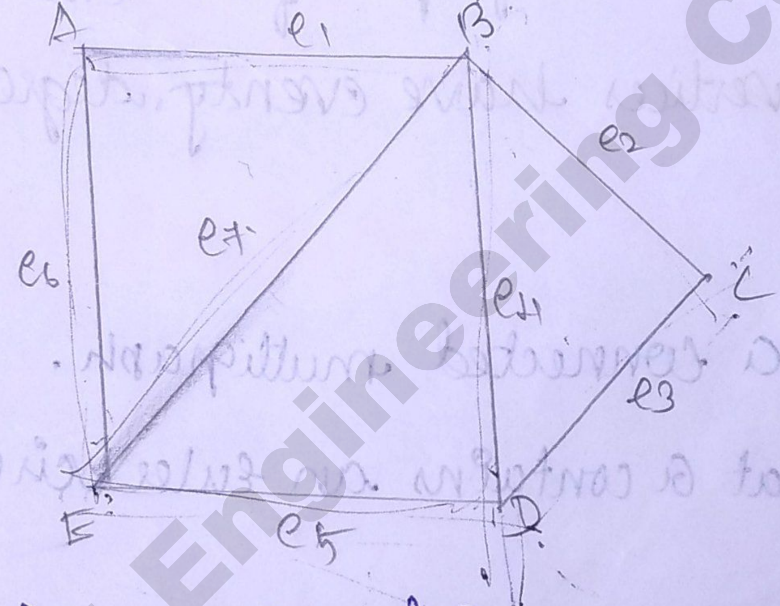
Hence the theorem.

Euler Graph:-

Eulerian path:-

A path of a graph G is called Eulerian path. If it contains each edge of the graph exactly once.

Example:-



Euler path between E and D

E - D - C - B - A - E - B - D.

E - A - B - C - D - B - E - D.

Eulerian Circuit: (or) Eulerian Cycle:-

An Eulerian circuit or Eulerian cycle should satisfy the following conditions.

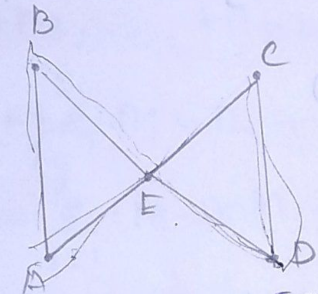
i) Starting & Ending vertices are same.

ii) Cycle should contain all the edges of Graph but exactly once.

Euler Graph:-

Any graph containing an Eulerian circuit or Eulerian cycle is called an Euler graph.

Ex:



A - E - D - C - E - B - A.

A connected multigraph of an Euler circuit iff and only if all vertices have evenly degree.

Proof:-

Let G be a connected multigraph.

Suppose that G contains an Euler circuit say from V_0 to V_0.

$$V_0 - e_1 - V_1 - e_2 \ldots\ldots V_{n-1} - e_n - V_0$$

Both edges e_1 and e_n contribute a 1 to deg (V_0) and so deg (V_0) is atleast two.

Each time the circuit passes through a vertex (Including V_0) the degree of vertex is increased by 2.

Consequently the degree of all vertices including V_0 is an ~~are~~ even.

Conversely,

Suppose that every vertex of G is ~~has~~ even degree.

The construction of the Euler circuit in G. proves G is an Euler graph.

Let V_0 be an arbitrary vertex in G.

Begining with V_0, form a circuit $C_1 : V_0 - V_1 - V_2 \cdots V_{n-1} V_0$

This is possible since every vertex has even degree and a vertex ($\neq V_0$) can be left by an edge not used to enter it.

If C_1 is Eulerian, G is an Eulerian Graph.

If C_1 not Eulerian, consider the subgraph H obtained by deleting all the edges in C_1 and vertices not incident with the remaining edges.

Note that all vertices of H have even degree. since G is connected H and C_1 must have a common vertex V.

Begining with V, construct a circuit C_2 for H. consider a larger circuit C by combining C_1 and C_2.

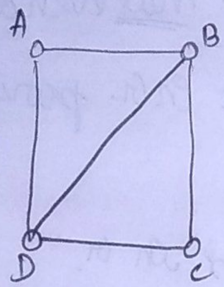
If C is Eulerian, G has an Euler circuit.

If C is not Eulerian, continue this procedure to ~~form an~~ forms Eulerian circuit. This procedure must terminate since G has finite no. of edges, ~~the process terminates~~ in G is finite.

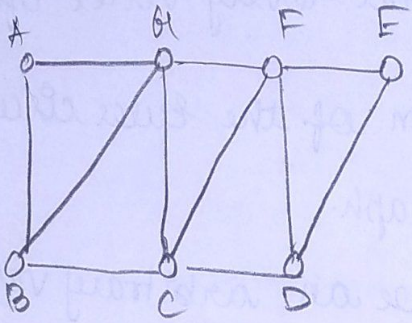
Thus G contains an Eulerian circuit & hence is Eulerian.

04/09/18

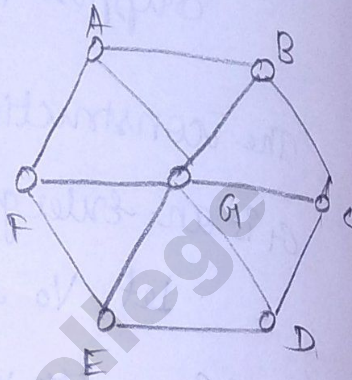
Find an euler path and an Eulerian circuit if it exists in each of the 3 Graphs. If it does not exist explain why?.



G_1



G_2.



G_3.

Sol:

1) In the graph G_1 the vertices B and D have odd degree namely 3.

∴ G_1 contains exactly two vertices (B & D) of odd degree.

Then by above result G_1 has an Eulerian path which have an end points as B and D. And does not have an Eulerian circuit.

Euler path:-

D—A-B-C-D-B

Since Eulerian circuit does not exist for G_1.

The given graph not Eulerian circuit.

2)

G_2 has exactly two vertices of odd degree namely B & D. So it has an Euler path. that must have B and D as endpoints and does not have an Eulerian circuit.

One such Euler path is

B — A — G — F — E — D — C — ~~B~~ G — B — C — F — D

Since G_2 does not have Eulerian circuit.

$\therefore$ G_2 is not Euler graph.

3) In G_3 there are 6 vertices of odd degree hence G_3 contains neither an Euler path nor Eulerian circuit.

3.61, P$_{\text{ob}}$ No: 10

Hamiltonian path:-

A path of a graph G is called a Hamiltonian path if it includes each vertex of G exactly once.

Hamiltonian Circuit:-

A circuit of a graph G is called an Hamiltonian circuit. If it includes each vertex of G exactly once except the starting and end vertices.

Hamiltonian Graph:-

Any graph containing a Hamiltonian circuit or cycle is called Hamiltonian graph.

Theorem 1:-

Show that complete graph on ~~end~~ 'n' vertices K_n with $n \geq 3$ has Hamiltonian cycle.

sol:

Let U be any vertex of K_n.

Since K_n is Complete graph with n vertices, any two vertices are joined.

So, We start with U and visit vertices in any order exactly once. and come back to U.

Hence there is a Hamiltonian cycle in K_n. and thus K_n is hamiltonian.

Theorem 2:-

If G is a connected simple graph. with n vertices with $n \geq 3$ such that the degree of every vertex in G is atleast $n/2$, then prove that G has Hamiltonian cycle.

Proof:-

Assume that the theorem is false.

And Let G be a maximal non-hamiltonian. Simple graph with $n \geq 3$ and the degree of each vertex is atleast $n/2$..

i.e,

$d(v) \geq n/2$ for all vertices in G.

Let U and V be not adjacent vertices in G.

Then $d(u) + d(v) \geq n/2 + n/2$.

$d(u) + d(v) \geq n$.

∴ For two not adjacent vertices U and V. the result holds

$d(u) + d(v) \geq n$

which is contradiction to the hypothesis that the degree of vertex is atleast $n/2$.

∴ G is hamiltonian cycle.

Give an example of a graph which is.

i) Eulerian but not hamiltonian.

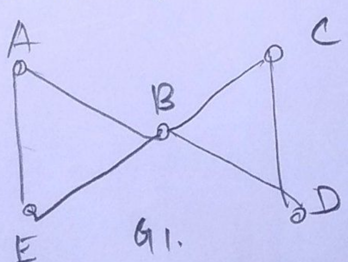
ii) Hamiltonian but not Eulerian.

iii) Both Hamiltonian & Eulerian.

iv) Both are not Hamiltonian or Eulerian.

Sol:

i) Example of Eulerian but not Hamiltonian graph.



G1.

Reason:-

G1 contains the Eulerian cycle is A-B-C-D-B-E-A (all the edges occurs exactly once, moreover the degree of all vertices in G1 is Even.)

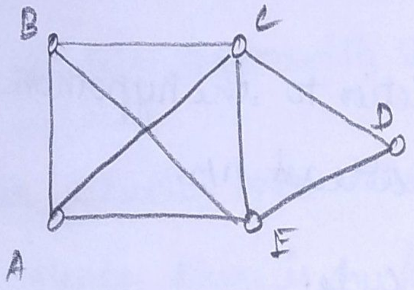
∴ G_1 is Euler Graph.

We cannot find Hamiltonian cycle as the vertex B is repeated twice.

∴ G_1 is not a Hamiltonian Graph.

Hence G_1 is Eulerian but not Hamiltonian.

2)



A - B - C - D - E - A.

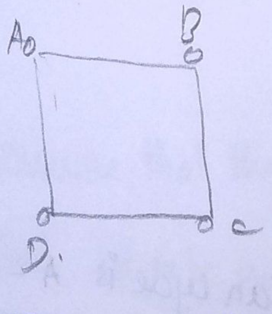
Since G_2 contains Hamiltonian cycle namely A-B-C-D-E-A. (all the Vertices occurs exactly once).

∴ G_2 is Hamiltonian Cycle.

Since the degree of vertex A is 3, d(A) is not an even.

∴ G_2 is not Eulerian.

3)



In G_3 consider the cycle A-B-C-D-A.

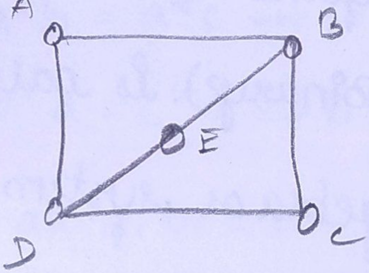
Since the cycle contains all the edges exactly once.

∴ G_3 is Eulerian.

And the cycle contains all the vertices exactly once.

$\therefore G_3$ is Hamiltonian.

4)



In G_4 degree of B equal to

$$deg(B) = deg(D) = 3.$$

Since the deg(B) and D are not even.

$\therefore G_4$ is not Euler Graph.

As no cycle passes through each of the vertices exactly once.

$\therefore G_4$ is not Hamiltonian.

Hence G_4 is neither Euler graph nor Hamiltonian.