# UNIT-V

18/09/18.

## LATTICES AND BOOLEAN ALGEBRA.

**Partial order relation:-**

A relation R on a set A is said to be a partial order relation. If R is reflexive, anti symmetric and transitive.

**Result:-**

Reflexive:- $xRx, \forall x \in R$

Anti-Symmetric:- $xRy$ & $yRx \Rightarrow x=y, \forall x,y \in R$.

Transitive:- $xRy$ & $yRz \Rightarrow xRz, \forall x,y,z \in R$.

**Poset (Partially Ordered set)**

A set P together with a partial order relation $\leq$ is called partially Ordered set or POSET and it is denoted by $(P, \leq)$

**Comparable property:-**

In a poset for any two elements $a, b$ either $a \leq b$ or $b \leq a$ is called comparable property.

**Totally ordered set:(or) Linearly Ordered set:(or) chain**

A partially Ordered set $(P, \leq)$ is said to be totally Ordered set. If any 2 elements are comparable.

(i.e.,)

Given any 2 elements ~~except~~ x and y of a poset either $x \leq y$ or $y \leq x$

Hasse ~~dialogue~~-gram:-

pictorial representation of a poset is called a Hasse diagram.

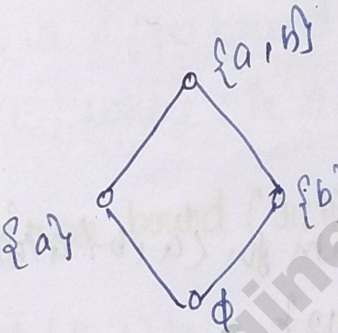
1. Draw a Hasse diagram for $(P(A), \leq)$ where
   i) $A = \{a, b\}$
   ii) $A = \{a, b, c\}$

Sol:

Hasse diagram for $(P(A), \leq)$

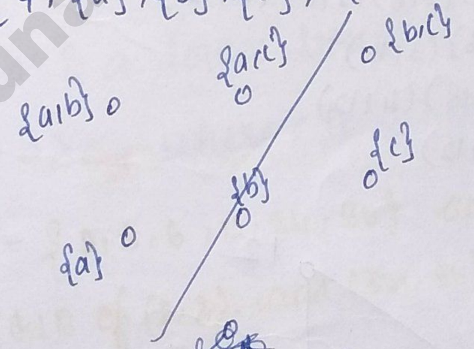
i) $A = \{a, b\}$

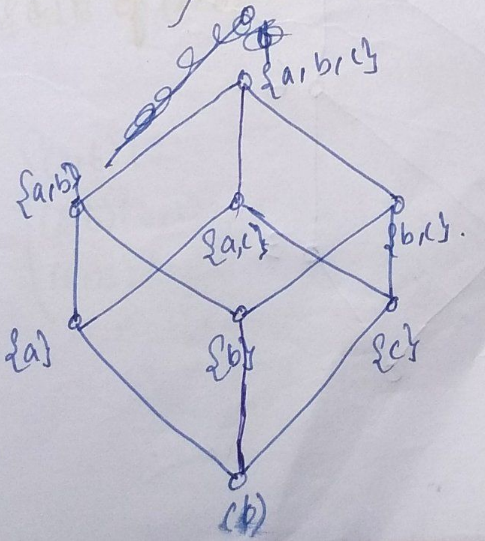
$R = \{\{a\}, \{b\}, \{a, b\}, \phi\}$




ii) $A = \{a, b, c\}$

$R = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\} \{a, b, c\}\}$

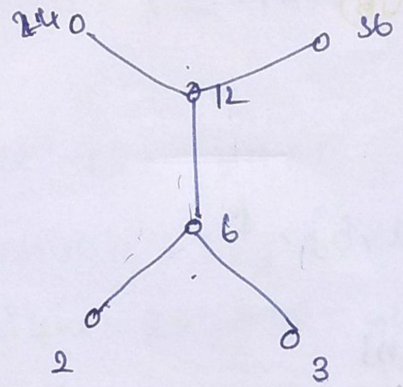


If $X = \{2, 3, 6, 12, 24, 36\}$ and the relation R defined on X by $R = \{\langle a, b\rangle / a \mid b\}$ Draw Hassee diagram for $X, R$.

Sol:

$$R = \left\{ \begin{matrix} (2,6)\ (2,12)\ (2,24)\ (2,36) \\ (3,6)\ (3,12)\ (3,24)\ (3,36) \\ (6,12)\ (6,24)\ (6,36) \\ (12,24)\ (12,36) \end{matrix} \right\}$$

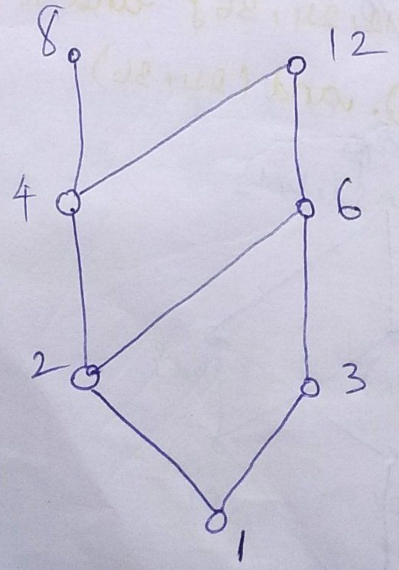



Draw the Hasse diagram for $\langle a, b / a \mid b\rangle$ on $\{1, 2, 3, 4, 6, 8, 12\}$

5·1
1
1 3+

5-8
1
2, 3, 4.

$$R = \left\{ \begin{matrix} (1,2)\ (1,3)\ (1,4)\ (1,6)\ (1,8)\ (1,12) \\ (2,4)\ (2,6)\ (2,8)\ (2,12) \\ (3,6)\ (3,12) \\ (4,8)\ (4,12) \\ (6,12) \end{matrix} \right\}$$



## Upper bound & Lower bound:-

Let $(P, \leq)$ be a poset and A be any non empty subset of P.

An element $a \in P$ is an upperbound of A. If $a \geq x \ \forall x \in A$.

An element $b \in P$ is said to be lowerbound for A if $b \leq x, \forall x \in A$.

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## Least upper bound:- (LUB)

Let $(P, \leq)$ be a poset and $A \subseteq P$. An element $a \in P$ is said to be least upper bound (LUB) of A if

i) 'a' is a upper bound of A.

ii) $a \leq c$, where c is anyother upper bound of A.

## Greatest lower bound (GLB):-

Let $(P, \leq)$ be a poset and $A \subseteq P$. An element $a \in P$ is said to be greatest lower bound (GLB) of A if

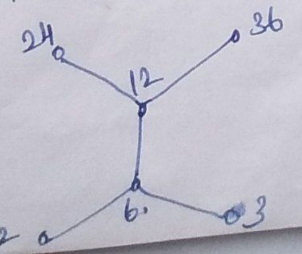
i) 'a' is a lower bound of A.

ii) $a \geq c$, where c is anyother lower bound of A.

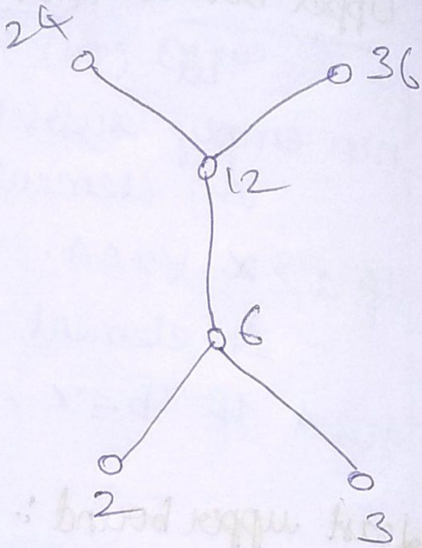
Consider $X = \{2, 3, 6, 12, 24, 36\}$ and $R = \{(a, b) / a / b\}$ find LUB & GLB of (2, 3). and (24, 36)

Sol:

$$R = \left\{ \begin{array}{l} (2,6)(2,12)(2,24)(2,36) \\ (3,6)(3,12)(3,24)(3,36) \\ (6,12)(6,24)(6,36) \\ (12,24)(12,36) \end{array} \right\}$$



1. UB $\{2,3\} = \{6, 12, 24, 36\}$




LUB $\{2,3\} = \{6\}$

2) UB $\{24, 36\}$ = does not exist

LUB $\{24, 36\}$ = does not exist

3) LB $\{2, 3\}$ = does not exist

GLB $\{2,3\}$ = does not exist.

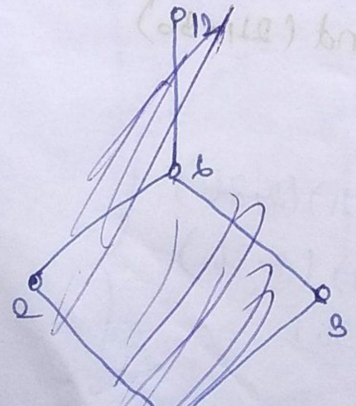
4) LB $\{24, 36\} = \{2, 3, 6, 12\}$

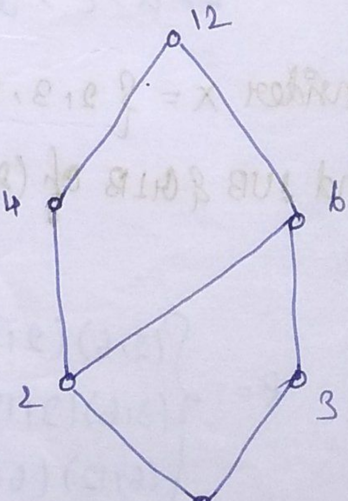
GLB $\{24, 36\} = \{12\}$

Consider $X = \{1, 2, 3, 4, 6, 12\}$ and $R = \{(a, b) / a/b\}$ find LUB & GLB. for $\{1, 3\}$ $\{1, 2, 3\}$, $\{2, 3\}$ $\{2, 3, 6\}$.

sol:

$$R = \begin{cases} (1,2)(1,3)(1,4)(1,6)(1,12) \\ (2,4)(2,6)(2,12) \\ (3,6)(3,12) \\ (4,12) \\ (6,12) \end{cases}$$



1) $UB\{1,3\} = \{3,6,12\}$

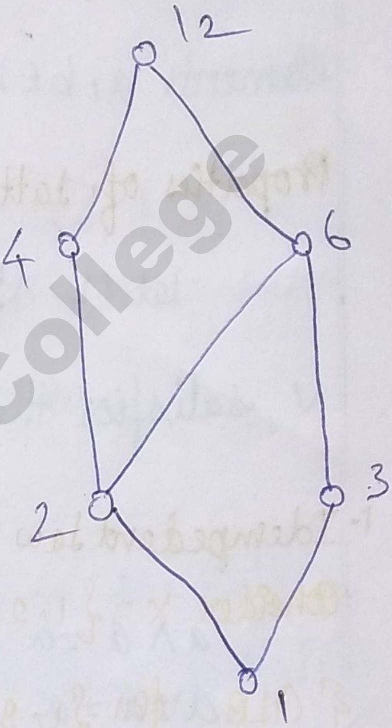
$LUB\{1,3\} = 3$

2) $UB\{1,2,3\} = \{6,12\}$

$LUB\{1,2,3\} = 6$

3) $UB\{2,3\} = (6,12)$

$LUB\{2,3\} = 6$




4) $LB\{1,3\} = 1$

$GLB\{1,3\} = 1$

5) $LB\{1,2,3\} = 1$

$GLB\{1,2,3\} = 1$

6) $LB\{2,3\} = 1$

$GLB\{2,3\} = 1$

7) $UB\{2,3,6\} = \{6,12\}$

$LUB\{2,3,6\} = 6$

8) $LB\{2,3,6\} = \{2,3,1\}$

$GLB\{2,3,6\} = 3$

$UB\{4,6\} = 12$

$LUB\{4,6\} = 12$

$LB\{4,6\} = \{1,2\}$

$GLB = \{2\}$.

Note:-

$GLB(a,b) = a * b$ (or) $a \wedge b$ (or) $a \cdot b$ [Meet]

$LUB(a,b) = a \oplus b$ (or) $a \vee b$ (or) $a + b$ [join]

Lattice:-

A Lattice is a poset $L, \leq$ in which for every pair of elements $a, b \in L$ both GLB and LUB exists.

Properties of Lattice:-

Let $(L, \wedge, \vee)$ be a given lattice then $\wedge$ and or $\vee$ satisfies the following conditions, $\forall\ a, b, c \in L$.

1. Idempedent Law:-

$a \wedge a = a$

$a \vee a = a$

2. Commutative Law:-

$a \wedge b = b \wedge a$

$a \vee b = b \vee a$

3. Associative Law:-

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

$(a \vee b) \vee c = a \vee (b \vee c)$

4. Absorption Law:-

$a \wedge (a \vee b) = a$

$a \vee (a \wedge b) = a$

5). $a \vee b = b$ iff $a \leq b$

$a \wedge b = a$ iff $a \leq b$

6) Consistency Law:-

$a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$.

Note:- 1) $(S_{24}, D)$ is a lattice

2) $(S_{30}, D)$ is a lattice

Meaning of $a \vee b$ and $a \wedge b$:-

i) $a \le a \vee b$ and $b \le a \vee b$

$\therefore$ $a \vee b$ is an upperbound of a and b.

ii) If $a \le c$ and $b \le c$ then $a \vee b \le c$

$\therefore$ $a \vee b$ is the LUB of a and b.

iii) $a \wedge b \le a$ and $a \wedge b \le b$

$\therefore$ $a \wedge b$ is lowerbound of a and b.

iv) If $c \le a$ and $c \le b$ then $c \le a \wedge b$.

$\therefore$ $a \wedge b$ is the GLB of a and b.

Meaning of Dual lattice:-

If $(L, \le)$ is a lattice then the lattice defined by $(L, \ge)$ where the partial ordering $\ge$ is the dual of the partial ordering $\le$ is called dual lattice.

(ie.,) $\wedge$ by $\vee$, $\vee$ by $\wedge$. is called dual statement.

Theorem:-

Let $(L, \le)$ be a distributive lattice then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c$. $\forall a, b, c \in L$.

(or)

cancellation property — Prove that $a \oplus b = a \oplus c$ and $a * b = a * c \Rightarrow b = c$.

Proof:-

W.K.T

$$
\begin{aligned}
b &= b \vee b \\
&= b \vee (a \wedge b) \\
&= b \vee (a \wedge c) \quad [\because a \wedge b = a \wedge c] \\
&= (b \vee a) \wedge (b \vee c)
\end{aligned}
$$

$= \phi$

$= (a \vee c) \wedge (b \vee c)$ [$\because a \vee b = a \vee c$].

$= (a \wedge b) \vee c$

$= (a \wedge c) \vee c = c$ [$\because$ Absorption law]

$\boxed{b = c}$

2) Theorem:-

State and prove distributive inequalities

Statement:-
Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$, then

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Proof:

W.K.T $a \wedge b \leq a$ and $a \wedge b \leq b \leq b \vee c$

$\therefore a \wedge b$ is a lower bound of $a$ and $b \vee c$

$\Rightarrow (a \wedge b) \leq a \wedge (b \vee c)$ — ①.

Again $a \wedge c \leq a$ and $a \wedge c \leq c \leq b \vee c$

$\therefore a \wedge c$ is a LB of $a$ and $b \vee c$

$\Rightarrow a \wedge c \leq a \wedge (b \vee c)$ — ②.

from ① & ②

$a \wedge (b \vee c)$ is an UB of $a \wedge b$ and $a \wedge c$.

But $(a \wedge b) \vee (a \wedge c)$ is the LUB of $a \wedge b$ and $a \wedge c$.

$\Rightarrow a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

By applying Duality,

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

3) Show that a chain is a lattice

Proof:-

Let $(L, \leq)$ be a chain

If $a, b \in L$, $a \leq b$ (or) $b \leq a$

If $a \leq b$, $a \wedge b = a$, $a \vee b = b$

$\therefore$ LUB and GLB of a and exists

If $b \leq a$, $b \wedge a = b$, $b \vee a = a$

$\therefore$ LUB and GLB of a and b exists

Hence every pair of elements has a GLB and LUB.

$\therefore$ Every chain is a lattice.

Distributive Lattice:-

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Lattice Homomorphism:-

Let $(L, *, \oplus)$ and $S, \wedge, \vee$ ~~with~~ be two lattices a mapping $g: L \rightarrow S$ is called lattice homomorphism. If $g(a * b) = g(a) \wedge g(b)$

$$g(a \oplus b) = g(a) \vee g(b)$$

Modular lattice:-

The lattice $L$ is said to be a modular lattice if $\forall a, b, c \in L$.

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

1) Theorem:-

prove that every distributive lattice is modular lattice.

proof:-

Let $(L, \leq)$ be a distributive lattice.

Let $a, b, c \in L$ such that $a \leq c$.

To prove: $L$ is modular lattice.

i.e, To prove: $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Now $a \leq c \Rightarrow a \vee c = c$.

L.H.S $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (a \vee b) \wedge c$

$= RHS$.

$\therefore L$ is a modular lattice.

2) Theorem:-

Show that every chain is a distributive lattice.

proof:-

Let $(L, \leq)$ be a chain.

If $a, b, c \in L$, then $a \leq b$ or $b \leq a$.

i) Suppose that

$a \leq b$ (or) $a \leq c$, then $a \leq b \vee c$

Let $a \wedge (b \vee c) = a \wedge a$

$= a$

$\because a \leq b \Rightarrow a \wedge b = a$

$a \leq c \Rightarrow a \wedge c = a$.

$a \leq b \vee c \Rightarrow a \wedge (b \vee c)$

R.HS

$(a \wedge b) \vee (a \wedge c)$

$= a \vee a$

$= a$

LHS = R.H.S

ii) ~~Suppose that~~

~~$b \leq a$ (or) $c \leq a$~~

By dualing, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

ii) suppose that

$b \le a$ (or) $c \le a$

then $b \vee c \le a$

Now

$b \le a \Rightarrow b \wedge a = b$

$c \le a \Rightarrow c \wedge a = c.$

$b \vee c \le a \Rightarrow (b \vee c) \wedge a = b \vee c$

L.H.S $a \wedge (b \vee c) = b \vee c$

RHS $(a \wedge b) \vee (a \wedge c) = b \vee c$

$\Rightarrow$ L.H.S = RHS

$\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

By dualing,

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Complement of an element:-

Let L be a bounded lattice with LB = 0 and UB = 1

$\therefore$ LB = 1 & UB = 0.

The element $x \in L$ is called the complement of ~~a~~ $a \in L$.

If $a \wedge x = 0$ and $a \vee x = 1$.

complemented lattice:-

A lattice L is said to be complemented if it is bounded and every element in it has atleast 1 complement.

## Bounded lattice:-

A Lattice L is said to be bounded if it has both LB and UB, if L is a bounded lattice then,

$a \vee 0 = a$, $a \wedge 0 = 0$

$a \vee 1 = 1$, $a \wedge 1 = a$.

**Theorem:-** (8m)

prove that in a bounded distributive lattice the complement of any element is unique.

**proof:-**

Let L be a bounded distributive lattice.

Let b & c be complements of an element $a \in L$.

To prove $b = c$.

Since b & c are complements of 'a' we have.

$a \wedge b = 0 \rightarrow (1)$, $a \vee b = 1 \rightarrow (2)$.

$a \wedge c = 0 \rightarrow (3)$, $a \vee c = 1 \rightarrow (4)$.

Now,

$$
\begin{aligned}
b &= b \wedge 1 \\
&= b \wedge (a \vee c) && \because \text{by } (4) \\
&= (b \wedge a) \vee (b \wedge c) \\
&= 0 \vee (b \wedge c) && \because \text{by } (1) \\
&= (a \wedge c) \vee (b \wedge c) && \because \text{by } (3) \\
&= (a \vee b) \wedge c && \because \text{by } (2) \\
&= 1 \wedge c = c
\end{aligned}
$$

$$\boxed{\therefore b = c}$$

$\therefore$ The complement of any element is unique.

## De morgan's Law:-

Prove de morgan's law in a bounded, complemented and distributive lattice.

**Sol:**

Demorgan's laws are given by

$(a \vee b)' = a' \wedge b'$ (or) $(a \oplus b)' = a' * b'$

$(a \wedge b)' = a' \vee b'$ (or) $(a * b)' = a' \oplus b'$

To prove: $(a \vee b)' = a' \wedge b'$

W.K.T $a \vee a' = 1 \rightarrow (1)$. $a \wedge a' = 0 \rightarrow (2)$

$b \vee b' = 1 \rightarrow (3)$ $b \wedge b' = 0 \rightarrow (4)$

and $1 \vee x = 1 \rightarrow (5)$. $0 \wedge x = 0 \rightarrow (6)$.

consider, $(a \vee b) \vee (a' \wedge b')$

$= (a \vee b \vee a') \wedge (a \vee b \vee b')$

$= (1 \vee b) \wedge (a \vee 1)$ $\because$ by (1) & (3).

$= 1 \wedge 1$ $\because$ by (5).

$(a \vee b) \vee (a' \wedge b') = 1 \rightarrow (7)$.

$(a \vee b) \wedge (a' \wedge b')$

Now $(a \vee b) \wedge (a' \wedge b')$

$= (a \wedge a' \wedge b') \vee (b \wedge a' \wedge b')$

$= (0 \wedge b') \vee (0 \wedge a')$ $\because$ by (2) & (4)

$= 0 \vee 0$ $\because$ by (6).

$= 0 \rightarrow (8)$.

from (7) & (8)

$\therefore a' \wedge b'$ is the complement of $a \vee b$

Hence $(a \vee b)' = a' \wedge b'$

By duality $(a \wedge b)' = a' \vee b'$.

## Boolean algebra:

A complemented distributive lattice is called a boolean algebra.

### Abstract definition:-

Let B be a non-empty set that contain two special elements 0 (element) and 1 (unity) and on which we defined closed binary operations '+', '·', and a unary operation '–' (complement) then $(B, +, -, 0, 1)$ is called a boolean algebra.

If the following conditions are satisfied

For $x, y, z \in B$

1. $x+y = y+x$, $x \cdot y = y \cdot x$.
2. $x+(y+z) = (x+y)+z$, $x \cdot (y \cdot z) = (x \cdot y) z$.
3. $x+0 = x$, $x \cdot 1 = x$.
4. $x+\bar{x} = 1$, $x \cdot \bar{x} = 0$.
5. $x+(y \cdot z) = (x+y) \cdot (x+z)$

   $x \cdot (y+z) = (x \cdot y)+(x \cdot z)$

### Laws of Boolean Algebra:

1. $x \cdot 0 = 0$ ; $x+1 = 1$ [Dominance law]
2. $x \cdot (x+y) = x$ ; $x+(x \cdot y) = x$.
3. $$\left.\begin{array}{l} \left.\begin{array}{l} xy = z \\ \bar{x}y = \bar{x}z \end{array}\right\} \Rightarrow y = z. \\ \left.\begin{array}{l} x+y = x+z \\ \bar{x}+y = \bar{x}+z \end{array}\right\} \Rightarrow y = z \end{array}\right\} \text{cancellation law.}$$
4. $x \cdot (yz) = (x \cdot y) z$.

   $x+(y+z) = (x+y)+z$.
5. $x+y = 1$ ; $x \cdot y = 0$

   $\Rightarrow x = \bar{y}$ (or) $y = \bar{x}$

v) $x+y=1$ ; $x\cdot y=0$

$x=\overline{y}$ (or) $y=\overline{x}$

vi) $\overline{\overline{x}}=x$.

vii) $(\overline{x+y})=\overline{x}\cdot\overline{y}$

$(\overline{x\cdot y})=\overline{x}+\overline{y}$

viii) $\overline{0}=1$ ; $\overline{1}=0$.

Show that demorgan's law are valid in boolean algebra.

Sol:

Demorgan's law are $(a+b)'=a'\cdot b'$.

$(a\cdot b)'=a'+b'$.

i) Consider $(a+b)+(a'\cdot b')$

$=(a+b+a')\cdot(a+b+b')$

$=(1+b)\cdot(a+1)$

$=1\cdot 1$

$=1$.

$(a+b)\cdot(a'\cdot b')=(a\cdot a'\cdot b')+(b\cdot a'\cdot b')$

$=(0\cdot b')+(a'\cdot 0)$

$=0+0$

$=0$.

$\therefore a'\cdot b'$ is the complement of $(a+b)$

Hence $(a+b)'=a'\cdot b'$.

ii) $(a\cdot b)+(a'+b')$

$=(a+a'+b')\cdot(b+a'+b')$

$=(1+b')\cdot(a'+1)$

$=1\cdot 1$

$=1$.

$(a\cdot b)\cdot(a'+b')=(a\cdot b\cdot a')+(a\cdot b\cdot b')$

$=(0\cdot b)+(a\cdot 0)$

$=0+0$

$=0$.

$\therefore a'+b'$ is the complement of $a\cdot b$

Hence $(a\cdot b)'=a'+b'$.

Theorem:-

If any boolean algebra, show that $(a+b')$

$(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

Sol:

<u>LHS</u>

$(a+b')(b+c')(c+a')$

$= (a+b')(bc + ba' + c'c + c'a')$

$= abc + aba' + ac'c + ac'a' + b'bc +$

$b'ba' + b'c'c + b'c'a'$

$= abc + 0 + 0 + 0 + 0 + 0 + 0 + 0 + b'c'a'$

$= abc + b'c'a'$ —— (1)

<u>RHS</u>

$(a'+b)(b'+c)(c'+a)$

$= (a'+b)(b'c' + b'a + cc' + ca)$

$= a'b'c' + a'b'a + a'cc' + a'ca + bb'c' + bb'a$

$+ bcc' + bca.$

$= a'b'c' + 0 + 0 + 0 + 0 + 0 + 0 + bca$

$= abc + a'b'c'$ —— (2)

from (1) & (2)

L.H.S = R.H.S.

$(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a).$

$a \le b \Leftrightarrow a \wedge \bar{b} = 0 \Leftrightarrow$

Show that in a boolean algebra,

$$a \le b \Leftrightarrow a \wedge \bar{b} = 0 \Leftrightarrow \bar{a} \vee b = 1 \Leftrightarrow \bar{b} \le \bar{a}.$$

Sol:

i) To prove;

$$a \le b \Rightarrow a \wedge \bar{b} = 0.$$

W.K.T $a \le b \Rightarrow a \wedge b = a \rightarrow (1)$ $\quad a \vee b = b \rightarrow (2).$

$$a \wedge \bar{b} = \underset{=(a \wedge b) \wedge \bar{b}}{a \wedge (b \wedge \bar{b})} \quad \because \text{by } (1)$$

$$= a \wedge 0$$

$$= 0$$

$$\therefore a \wedge \bar{b} = 0$$

ii) To prove.

$$a \wedge \bar{b} = 0 \Rightarrow \bar{a} \vee b = 1.$$

Now $a \wedge \bar{b} = 0.$

Taking complement on b/s.

$$\overline{a \wedge \bar{b}} = \bar{0}$$

$$\therefore \bar{a} \vee b = 1.$$

iii) to prove: $\bar{a} \vee b = 1 \Rightarrow \bar{b} \le \bar{a}$

Now, $\bar{a} \vee b = 1$

$$(\bar{a} \vee b) \wedge \bar{b} = 1 \wedge \bar{b}$$

$$(\bar{a} \wedge \bar{b}) \vee (b \wedge \bar{b}) = 1 \wedge \bar{b}$$

$$(\bar{a} \wedge \bar{b}) \vee 0 = \cancel{1 \wedge \bar{b}}\ \bar{b}.$$

$$\bar{a} \wedge \bar{b} = \bar{b}$$

$$\bar{b} \le \bar{a}$$

iv) To prove:

$$\bar{b} \le \bar{a} \Rightarrow a \le b$$

$$\bar{b} \le \bar{a} \Rightarrow \bar{a} \wedge \bar{b} = \bar{b}$$

$$\Rightarrow \overline{a \vee b} = \bar{b}$$

$$\Rightarrow a \vee b = b$$

$$\Rightarrow a \le b.$$

No: 5. 44.

In a boolean algebra show that the following statements are equivalent. For any a and b.

i) $a+b=b$ ii) $a.b=a$ iii) $a'+b=1$. iv) $a.b'=0$

v) $a\leq b$

Sol:

i) $\Rightarrow$ ii:

Assume that $a+b=b$.

$$a=a.(a+b)$$
$$\boxed{=a.b=a}$$

ii) $\Rightarrow$ iii:

Assume that $a.b=a$

$$a'+b=(a.b)'+b$$
$$=a'+b'+b$$
$$=a'+1$$
$$\boxed{a'+b=1}$$

iii $\Rightarrow$ iv:

Assume that $a'+b=1$

$$(a'+b)'=(1)'=0$$
$$a.b'=0$$

iv $\Rightarrow$ i:

Assume that $a\leq b$

~~a+b~~. $a+b=b$ and $a.b=a$

$$=a.b+1.b$$
$$=(a+1).b$$
$$=1.b$$
$$=b.$$

iv $\Rightarrow$ v: Assume that $a.b'=0$

then $a.b=a.b+0$

$$=a.b+ab'$$
$$=a.(b+b')$$
$$=a.1$$
$$a.b=a. \quad =a\leq b.$$