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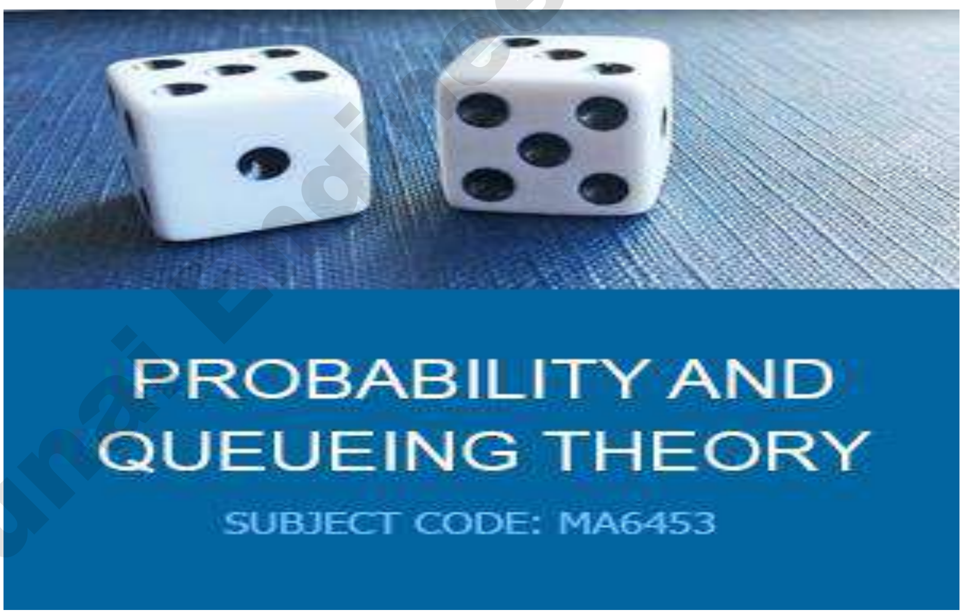
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

BACHELOR OF ENGINEERING

THIRD YEAR

Sixth Semester



MA6453 - Probability & Queueing Theory

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MA 6453 PROBABILITY AND QUEUEING THEORY L T P C

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UNIT I RANDOM VARIABLES 9+3

Discrete and continuous random variables – Moments – Moment generating functions – Binomial, Poisson, Geometric, Uniform, Exponential, Gamma and Normal distributions.

UNIT II TWO- DIMENSIONAL RANDOM VARIABLES 9+3

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and Linear regression – Transformation of random variables.

UNIT III RANDOM PROCESSES 9+3

Classification – Stationary process – Markov process - Poisson process – Discrete parameter Markov chain – Chapman Kolmogorov equations – Limiting distributions.

UNIT IV QUEUEING MODELS 9+3

Markovian queues – Birth and Death processes – Single and multiple server queueing models – Little's formula - Queues with finite waiting rooms – Queues with impatient customers: Balking and reneging.

UNIT V ADVANCED QUEUEING MODELS 9+3

Finite source models - M/G/1 queue – Pollaczek Khinchin formula - M/D/1 and M/EK/1 as special cases – Series queues – Open Jackson networks.

UNIT-1

19/12/18 Probability and Queueing Theory.

Unit-1 :-

Random Variable.

A random variable may be defined as a rule or fn's that assign real no to each possible outcomes of the random.

$X(S) =$ number of heads.

coin tossing twice

$S = \{HH, HT, TH, TT\}$

Discrete Random Variable :-

The 'x' be a random variable which can take a finite no. or countably infinite no of values. Then x is called discrete Random Variable.

Continuous Random Variable :-

The 'x' be a random variable which can take all values (infinite no of values) in and interval than x is called CRV.

Discrete Random Variable:-

probability Mass function or probability function [PMF]

Let X be a Discrete random Variable take the values $x_1, x_2, x_3, \dots$ such that $P(X=x_i) = P(x_i) = P_i$ where $i = 1, 2, \dots$ then $P(x_i)$ is called PMF satisfying the following condition

i) $P(x_i) \geq 0$

ii) $\sum_{i=1}^{\infty} P(x_i) = 1$.

Distribution function (or) Commulative D.f.

Let X be a random Variable then $P(X \leq x)$ is called the DF of the X it is denoted by $F(x) = P(X \leq x)$.

X is discrete

$$F(x) = \sum_i P(x_i)$$

X is continous

$$F(x) = \int_{-\infty}^{x} f(x)\, dx.$$

Continuous Random Variable / Probability density fn.

Let x be a continuous fn then f(x) is PDF satisfying the following condition

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

Properties of commulative distribution fn :-

i) $0 \leq F(x) \leq 1$

ii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

iii) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

iv) If $x \leq y \Rightarrow F(x) \leq F(y)$

① A Discrete Random Variable X for the following Probability distributions :-

x	0	1	2	3	4	5	6	7	8
$P(X=x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

1) Find a

2) Find $P(X<3)$ and $P(0<X<3)$

3) $P(X \geq 3)$ 4) Find the DF / CDF of X

i) WKT.

$$\sum_{i=1}^{n} P(x_i) = 1$$

$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$81a = 1$

$$\boxed{a = \frac{1}{81}}$$

ii) $P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81}$

$= \frac{9}{81} = \frac{1}{9}.$

iii) $P(0<X<3)$

$= P(1) + P(2) = 8a$

$= \frac{8}{81}$

iii) $P(X \geq 3) = 1 - P(X<3)$

$= 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore P(X \geq a) = 1 - P(X<a)$

iv) CDF

x	$F(x) = P(X \leq x)$
0	$F(0) = P(0) = a = \frac{1}{81}$
1	$F(1) = P(1) = 4a = \frac{4}{81}$
2	$F(2) = P(0) + P(1) + P(2) = 9a = \frac{9}{81}$

3)	$F(3) = 16a = \frac{16}{81}$
4)	$F(4) = 25a = \frac{25}{81}$
5	$F(5) = 36a = \frac{36}{81}$
6	$F(6) = 49a = \frac{49}{81}$
7	$F(7) = 64a = \frac{64}{81}$
8	$F(8) = 81a = \frac{81}{81} = 1$

2)

x	0	1	2	3	4
$P(X=x)$	k	3k	5k	7k	9k

i) Find k ii) Find $P(X<3)$ 3) $P(X\geq 3)$ & $P(0<X<4)$

④ Find DF or CDF of X.

i)

$\Rightarrow k+3k+5k+7k+9k = 1$

$25k = 1$

$k = \frac{1}{25}$

ii) $P(X<3)$

$= P(0) + P(1) + P(2)$

$= \frac{1}{25} + \frac{3}{25} + \frac{5}{25}$

$= \frac{9}{25}$

$P(0 < X < 4)$

$= P(1) + P(2) + P(3)$

$= 15k = \frac{15}{25}$

3) $P(X \geq 3) = 1 - P(X < 3)$

$= 1 - \frac{9}{25}$

$= \frac{16}{25}$

4)

x	$F(x) = P(X \leq x)$
0	$F(0) = P(0) = k = \frac{1}{25}$
1	$F(1) = P(0) + P(1) = 4k = \frac{4}{25}$
2	$F(2) = P(0) + P(1) + P(2) = 9k = \frac{9}{25}$
3	$F(3) = P(0) + P(1) + P(2) + P(3) = 16k = \frac{16}{25}$
4	$F(4) = 25k = 1.$

3) A random variable X has the following probability fn.

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Find k ii) $P(x<6)$ iii) $P(x \geq 6)$
iv) $P(0<x<5)$ v) If $P(x \leq c) > \frac{1}{2}$ find the minimum
vi) DF.

WKT.

$$\sum_{x=0}^{n} P(x_i) = 1$$

i) $\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$$10k^2+9k-1 = 0.$$

$$10k^2+10k-k-1=0$$

$$10k(k+1)-1(k+1)=0$$

$$(k+1)(10k-1)=0$$

$$k=-1 \; ; \; k=\frac{1}{10}$$

We take $k=\frac{1}{10}$

Now table,

x	0	1	2	3	4	5	6	7
$P(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

ii) $P(x<6)$

$= P(0)+P(1)+P(2)+P(3)+P(4)+P(5)$

$= 0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}$

$= \frac{8}{10}+\frac{1}{100}$

$= \frac{80+1}{100} = \frac{81}{100}$.

3) $P(X \geq 6)$

$= 1 - P(x < 6)$

$= 1 - \frac{81}{100}$

$= \frac{100-81}{100}$

$= \frac{19}{100}$.

4) $P(0 < x < 5)$

$= P(1) + P(2) + P(3) + P(4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$= \frac{8}{10}$

5)

x	$F(x) = P(X \leq x)$
0	$F(0) = P(0) = 0$
1	$F(1) = P(0) + P(1) = \frac{1}{10}$
2	$F(2) = P(0) + P(1) + P(2) = \frac{3}{10}$
3	$F(3) = \frac{5}{10} = \frac{1}{2}$
4	$F(4) = 8/10$

5 $F(5) = \frac{81}{100}$

6 $F(6) = \frac{83}{100}$

7 $F(7) = \frac{100}{100} = 1$

5) Minimum value of c.

$P(X \leq c) \geq \frac{1}{2}$

$C = 4$

The random variable X take the value 1, 2, 3, 4 ST $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution.

$P(X=1) = \frac{K}{2}$

$P(X=2) = \frac{K}{3}$

$P(X=3) = K$

$P(X=4) = \frac{K}{5}$

$K = \frac{30}{61}$

x	0	1	2	3	4
P(x)	0	K/2	K/3	K	K/5

$$\sum_{x=0}^{n} P(x_i) = 1$$

$$0 + \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{3k}{6} + \frac{2k}{6} + \frac{5k}{5} + \frac{k}{5} = 1 \Rightarrow \frac{5k}{6} + \frac{6k}{5} = 1$$

$$\frac{25k}{30} + \frac{36k}{30} = 1 \Rightarrow \frac{61k}{30} = 1$$

$$\boxed{k = \frac{30}{61}}$$

ii) CDF:-

$$F(1) = P(1) = \frac{30/61}{2} = \frac{30}{122} = 0.24$$

$$F(2) = P(1) + P(2) = 0.24 + \frac{30/61}{3} = 0.24 + \frac{30}{183} = 0.40$$

$$F(3) = P(1) + P(2) + P(3) = 0.24 + 0.40 + 30/61 =$$

$$F(4) = P(1) + P(2) + P(3) + P(4) =$$

Formula :-

1) $P(X \le x) = 1 - P(X > x)$

2) $P(X > x) = 1 - P(X \le x)$

3) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

4) $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx.$

5) $f(x) = \frac{d}{dx} F(x)$

6) $E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx.$

7) $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$.

↓ mean

8) $E(x) = \sum x_i P(x_i)$

9) $E(x^2) = \sum x_i^2 P(x_i)$

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{o.w} \end{cases}$$

i) find c ii) Find $P[X>1]$

i) WKT

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

$$\Rightarrow \int_{-\infty}^{0} + \int_{0}^{2} + \int_{2}^{\infty} = 1$$

$$\Rightarrow \int_{0}^{2} c(4x - 2x^2)\, dx = 1.$$

$$\Rightarrow c\left[4\frac{x^2}{2} - 2\frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow c\left[8 - \frac{16}{3}\right] = 1$$

$$c\left[\frac{24-16}{3}\right] = 1$$

$$c\left[\frac{8}{3}\right] = 1$$

$$c = \frac{3}{8}$$

ii) $P(X>1) = \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx$

$= \frac{3}{8}\left[4\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right) \right]_1^2$

$= \frac{3}{8}\left[2(4) - 2\left(\frac{8}{3}\right) \right] - \left[2(1) - \frac{2}{3} \right]$

$= \frac{3}{8}\left[8 - \frac{16}{3} - 2 + \frac{2}{3} \right]$

$= \frac{3}{8}\left[6 - \frac{14}{3} \right]$

$= \frac{3}{8}\left(\frac{4}{3}\right)$

$= \frac{1}{2}.$

Continous fn assume the value of $x=2$ & $x=5$. The density fn $f(x) = k(1+x)$. Find $P(x<4)$

$\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$\Rightarrow \int_{-\infty}^{0} + \int_{2}^{5} + \int_{2}^{\infty} = 1.$

$\Rightarrow \int_2^5 k(1+x)\,dx = 1$

$\Rightarrow k\left[x+\frac{x^2}{2}\right]_2^5=1$

$\Rightarrow \left[k\left(5+\frac{25}{2}\right)-\left[2+\frac{4}{2}\right]\right)=1$

$\Rightarrow k\left[\left(\frac{35}{2}\right)-\frac{8}{2}\right]=1$

$\Rightarrow k\left(\frac{27}{2}\right)=1$

$\Rightarrow \quad k=\frac{2}{27}.$

ii) $P(x\leq 4)=\int_2^4 \frac{2}{27}(1+x)\,dx$

$\frac{16}{27}$

$=\frac{2}{27}\int_2^4 (1+x)\,dx$

$=\frac{2}{27}\left[x+\frac{x^2}{2}\right]_2^4$

$=\frac{2}{27}\left[4+\frac{16}{2}\right]-\left[2+\frac{4}{2}\right]$

$=\frac{2}{27}\left[\frac{24}{2}-\frac{8}{2}\right]$

$=\frac{2}{27}\left[\frac{16}{2}\right]=\frac{16}{27}$

x	-2	-1	0	1	2	3
$P(x)$	0.1	k	0.2	$2k$	0.3	$3k$

i) find k ii) $P(x<2)$ iii) $P(-2<x<2)$
iv) DF.

i) $0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$6k + 0.6 = 1$

$6k = 1 - 0.6$

$6k = 0.4$

$k = \frac{0.4}{6}$

$k = \frac{1}{15}$

ii) $P(x<2) = P(0) + P(1) + P(-1) + P(-2)$

$= 0.2 + 2k + k + 0.1$

$= 0.3 + 3k$

$= 0.3 + \frac{1}{15} = \frac{7.5}{15} = 0.5$

iii) $P(-2<x<2)$

$= P(-1) + P(0) + P(1)$

$= k + 0.2 + 2k$

$= \frac{1}{15} + 0.2 + \frac{2}{15}$

$= \frac{3}{15} + 0.2$

$= \frac{16}{15} \Rightarrow 0.4$

iv) DF

$F(x) = P(X \le x)$

$F(-2) = P(-2) = 0.1$

$F(-1) = P(-2) + P(-1) = 0.1 + k = \frac{1.5}{15} + \frac{1}{15} = \frac{2.5}{15}$

$F(0) = P(-2) + P(-1) + P(0) = 0.3 + k$

$= 0.1 + \frac{1}{15} + 0.2 = 5.5/15$

$F(1) = P(-2) + P(-1) + P(0) + P(1) = 0.1 + \frac{1}{15} + 0.2 + \frac{2}{15}$

$= 7.5/15$

$F(2) = 0.1 + k + 0.2 + 2k + 0.3 = 0.1 + \frac{1}{15} + 0.2 + \frac{2}{15}$

$+ 0.3 = 12/15$

$F(3) = 0.1 + k + 0.2 + 2k + 0.3 + 3k$

$= 0.1 + \frac{1}{15} + 0.2 + \frac{2}{15} + 0.3 + \frac{3}{15} = 1.$

A Continuous Random Variable X

$$f(x) = \begin{cases} \frac{k}{1+x^2} & -\infty < x < \infty \\ 0 & \text{o.w} \end{cases}$$

i) find k. ii) D.F iii) $P(X \ge 0)$

i) Sln WKT

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$k \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1 \qquad \left[\because \int \frac{dx}{1+x^2} = \tan^{-1}(x)\right]$$

$$\Rightarrow k\left[\tan^{-1}(x)\right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow k\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = 1$$

$\therefore \left[\tan\frac{\pi}{2} = \infty \quad \frac{\pi}{2} = \tan^{-1}(\infty)\right]$

$$k\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

$$k(\pi) = 1$$

$$\boxed{k = \frac{1}{\pi}}$$

ii)

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{x} \frac{dx}{1+x^2}\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x) - \tan^{-1}(-\infty)\right]$$

$$F(x) = \frac{1}{\pi}\left[\tan^{-1}(x) + \pi/2\right]$$

iii)

$$P(x>0) = \int_{0}^{\infty} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{0}^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}.$$

If the density fn of CRV is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & OW \end{cases}$$

i) find a

ii) CDF of a.

WKT

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \left[\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{3} + \int_{3}^{\infty}\right]$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3ax - a\frac{x^2}{2}\right]_2^3 = 1$$

$$a\left[\frac{1}{2}\right] + a[1] + \left[\left[9a - \frac{9a}{2}\right] - (6a - 2a)\right] = 1$$

$$\frac{a}{2} + a + \frac{9a}{2} - 4a = 1$$

$$\frac{10a}{2} + a - 4a = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

To find DF

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 1 \\ \frac{1}{2} & 1 \le x \le 2 \\ \frac{3}{2} - \frac{x}{2} & 2 \le x \le 3 \\ 0 & \text{OW} \end{cases}$$

When x lies in $0 \le x \le 1$ (or) $[0,1]$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{0}^{x} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{4}$$

When x lies in $1 \le x \le 2$ (or) $[1,2]$

$$F(x) = \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{x}$$

$$= \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[x\right]_1^x$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left[x-1\right]$$

$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$

$F(x) = \frac{x}{2} - \frac{1}{4}$

When x lies in $2 \le x \le 3$

$F(x) = \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{x}$

$= \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^x f(x)dx$

$= \int_0^1 \frac{x}{2}dx + \int_1^2 \frac{1}{2}dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$

$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^2 + \left[\frac{3}{2}x - \frac{1}{2}\frac{x^2}{2}\right]_2^x$

$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(1) + \left[\frac{3}{2}x - \frac{x^2}{4}\right] - (3-1)$

$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 2$

$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$

$\frac{1}{4} + \frac{1}{2} - 2$

$= \frac{1+2-8}{4}$

$= \frac{-5}{4}$

When $x \ge 3$

$F(x) = \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{3} + \int_{3}^{\infty}$

$= \int_0^1 \frac{x}{2}dx + \int_1^2 \frac{1}{2}dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx$

$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^2 + \left[\frac{3}{2}x - \frac{1}{2}\left(\frac{x^2}{2}\right)\right]_2^3$

$$= \left[\frac{1}{4} + \frac{1}{2}[1] + \left(\frac{9}{2} - \frac{9}{4}\right) - (3-1)\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{9}{4} - 2.$$

$$= \frac{1+2+9-8}{4} = \frac{4}{4} = 1$$

2) $$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

Where x lies in $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} + \int_{0}^{x} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{x^2}{2}.$$

Where x lies in $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{x}$$

$$= \int_{0}^{1} x\,dx + \int_{1}^{x} 2-x\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_0^1 + \left[2 - \frac{x^2}{2}\right]_1^x$$

$$= \left[2 - \frac{1}{2}\right] + \left[2 - \frac{x^2}{2}\right] - \left[2 - \frac{1}{2}\right]$$

$$= \frac{1}{2} + \left[2x - \frac{x^2}{2} - 2 + \frac{1}{2}\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

When $x > 2$

$$F(x) = \int_{-\infty}^{0} + \int_0^1 + \int_1^2 + \int_2^x$$

$$= \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \left[4 - \frac{4}{2} - 2 + \frac{1}{2}\right]$$

$$= \frac{1}{2} + \left[4 - 2 - 2 + \frac{1}{2}\right]$$

$$= \frac{1}{2} + \left[4 - 4 + \frac{1}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1.$$

Problem based on Moments and Moments generating fn [MGF]. Mean & Variance.

The Cummulative Distribution fn of the random variable $F(x) = 1-(1+x)e^{-x}$, $x>0$. find the Mean and Variant.

We know,

$$f(x) = \frac{d}{dx}[F(x)]$$

$$= \frac{d}{dx}[1-e^{-x}-xe^{-x}]$$

$$= [0+e^{-x} - (x(-e^{-x}) + e^{-x}(1))]$$

$$= xe^{-x}$$

Mean

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x(xe^{-x}) dx$$

$$= \int_0^{\infty} \underset{u}{x^2} \underset{dv}{e^{-x} dx}$$

$$\left[\because \int u\,dv = uv - u'v_1 + u''v_2 + u'''v_3 + \ldots\right]$$

$$= \left[(x^2)(-e^{-x}) - (2x)(e^{-x}) + 2[-e^{-x}]\right]_0^{\infty}$$

$$= [0-[-2(1)]] = 2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^{\infty} x^2 (xe^{-x}) dx$$

$$= \int_0^{\infty} \underset{u}{x^3} \underset{dv}{e^{-x} dx}$$

$$= \left[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^\infty$$

$$= [0-(-6)] = 6.$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 6-(2)^2 = 6-4$$

$$=2$$

MGF

1) $M_X(t) = E[e^{tx}]$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \rightarrow \text{continuous case}$$

$$= \sum_{x=-\infty}^{\infty} e^{tx} p(x) \rightarrow \text{discrete case}$$

2) $\mu_r' = \frac{d}{dt}[M_X(t)]_{t=0}$

3) $M_X(t) = 1 + \frac{t}{\underline{|1}}\mu_1' + \frac{t^2}{\underline{|2}}\mu_2' + \frac{t^3}{\underline{|3}}\mu_3' + \dots + \frac{t^r}{\underline{|r}}\mu_r'$

Find the random variable X Then

$$f(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x>0 \\ 0 & o.w \end{cases}$$

i) Find MGF ii) Mean & Variance.

WKT

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{0} + \int_{0}^{\infty} e^{tx} \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-\left[\frac{1}{2}-t\right]x}\,dx$$

$e^{-\infty} = 0$

$e^{0} = e^{-0} = 1$

$$= \frac{1}{2}\left[\frac{e^{-\left[\frac{1}{2}-t\right]x}}{-\left(\frac{1}{2}-t\right)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[0 - \frac{1}{\left[-\frac{1}{2}+t\right]}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{1-2t}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-2t}\right]$$

$$= (1-2t)^{-1}$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+x^3+\dots\right]$$

$$= 1+2t+(2t)^2+(2t)^3+\dots$$

$$= 1+2\frac{t}{\lfloor 1} + 8\frac{t^2}{\lfloor 2} + \dots$$

$\mu_1' = 2, \quad \mu_2' = 8$

Variance

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= 8 - (2)^2$$

$= 8-4 = 4$

Find the MGF of the random variable having the pdf $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x>0 \\ 0 & \text{o.w} \end{cases}$ also reduce the 1st four moment about the origin.

WKT

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx}\left(\frac{x}{4} e^{-x/2}\,dx\right)$$

$$= \frac{1}{4}\int_0^{\infty} x\, \underbrace{e^{-\left[\frac{1}{2}-t\right]x}\,dx}_{dv}$$

$$= \frac{1}{4}\left[x\left(\frac{e^{-\left(\frac{1}{2}-t\right)x}}{-\left(\frac{1}{2}-t\right)}\right) - (1)\,\frac{e^{-\left(\frac{1}{2}-t\right)x}}{\left(\frac{1}{2}-t\right)^2}\right]_0^{\infty}$$

$$= \frac{1}{4}\left[0 - \left(-\frac{1}{\left(\frac{1}{2}-t\right)^2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{\frac{(1-2t)^2}{4}}\right] = \frac{1}{4}\left[\frac{4}{(1-2t)^2}\right]$$

$$M_X(t) = \frac{1}{(1-2t)^2}$$

$$\therefore (1-x)^{-2} = 1+2x+3x^2+4x^3+5(x)^4+\dots$$

$$= 1+2(2t)+3(2t)^2+4(2t)^3+5(2t)^4$$

$$= 1+\frac{4t}{1!}+\frac{12t^2}{2!}+\frac{32\times 6}{3!}t^3+\frac{5\times 16\times 24}{4!}t^4$$

$\mu_1' = 2 \quad \mu_2' = 9$

$$= 1 + \frac{4t}{\lfloor 1} + \frac{24t^2}{\lfloor 2} + \frac{192t^3}{\lfloor 3} + \frac{1920t^4}{\lfloor 4}$$

$\mu_1' = 4$ $\quad \mu_3' = 192$

$\mu_2' = 24$ $\quad \mu_4' = 1920$

Binomial Distribution:-

A random variable X is said to be Binomial distribution It assume only non-negative values of its probability mass fn.

Find the MGF Mean & Variance of the Binomial distribution.

$$P(X=x) = P(x) = \begin{cases} nC_x p^x q^{n-x} & x = 0, 1, 2, \dots n \\ 0 & \text{o.w} \end{cases}$$

Sln

To find MGF:

$P(X=x) = nC_x p^x q^{n-x}$, $x = 0, 1, 2, \dots n$

$M_x(t) = E[e^{tx}]$

$$= \sum_{x=0}^{n} e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} [nC_x p^x q^{n-x}]$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$\left[\because (p+q)^n = \sum_{x=0}^{n} nC_x p^x q^{n-x}\right]$$

$$M_X(t) = (pe^t + q)^n$$

To find Mean

$$\frac{d}{dt}[M_X(t)] = n[pe^t+q]^{n-1}(pe^t) = np((pe^t+q)^{n-1}e^t)$$

$$\frac{d^2}{dt^2}[M_X(t)] = np[(pe^t+q)^{n-1}(e^t) + (e^t)(n-1)(pe^t+q)^{n-2}(pe^t)]$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= np(1) = np$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= np[1+(n-1)p] = np[1+np-p]$$

$\therefore$ [$d(uv) = uv' + vu'$, $e^0 = 1$

$q = 1-p \Rightarrow p+q = 1.$]

$$= np + n^2p^2 - np^2$$

$$E(X^2) = np(1-p) + n^2p^2 = npq + n^2p^2$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= npq + n^2p^2 - n^2p^2$$

$$= npq.$$

~~Binomial~~, Poisson Distribution ~~Geometric~~

A random variable x is to be a Poisson ~~Passion~~ distribution if it assume non-negative values & its probability mass fn.

2] $P(X=x) = P(x) = \frac{e^{-\lambda} \lambda^{x}}{\lfloor x}$ $x = 0, 1, 2, \ldots \infty$

Soln

To find MGF.

$M_X(t) = E[e^{tx}]$

$= \sum_{x=0}^{\infty} e^{tx} P(x)$

$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^{x}}{\lfloor x}$

$= e^{-\lambda} \sum_{x=0}^{n} \frac{(\lambda e^{t})^{x}}{\lfloor x}$

* $= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{\lfloor 1} + \frac{(\lambda e^{t})^{2}}{\lfloor 2} + \frac{(\lambda e^{t})^{3}}{\lfloor 3} + \ldots\right]$

$= e^{-\lambda}[e^{\lambda e^{t}}] = e^{\lambda(e^{t}-1)}$

To find Mean.

$\frac{d}{dt}[M_X(t)] = e^{-\lambda}\left[e^{\lambda e^{t}}(\lambda e^{t})\right]$

$= \lambda e^{-\lambda}\left[e^{\lambda e^{t}}(e^{t}) + e^{t} e^{\lambda e^{t}}(\lambda e^{t})\right]$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + e^{\lambda}\lambda\right] = \lambda e^{-\lambda}(e^{\lambda})$$

$$= \lambda e^{-\lambda} e^{\lambda}[1+\lambda] = \lambda = \text{Mean.}$$

$$= \lambda + \lambda^2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda e^{\lambda}\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}[1+\lambda] = \lambda + \lambda^2$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda.$$

Geometric distribution.

A random variable X is said to be a Geometric distribution if it assume non-negative values & its probability Mass fn.

$$P(X=x) = P(x) = q^{x-1}p, \quad x = 1,2,3,\dots \quad 0 < p \le 1$$

where $q = 1-p$

[or]

$$P(X=x) = P(x) = q^{x}p, \quad x = 0,1,2,\dots \quad 0 < p \le 1$$

where $q = 1-p$

To find MGF.

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} (q^x p)$$

$$= p \sum_{x=0}^{\infty} (qe^t)^x$$

$$= p[1 + (qe^t) + (qe^t)^2 + (qe^t)^3 + \cdots]$$

$$[\because (1-x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= p[1-qe^t]^{-1}$$

$$= \frac{p}{[1-qe^t]} = p[1-qe^t]^{-1}$$

$$\frac{d}{dt} M_X(t) = p(-1)(1-qe^t)^{-2}(-qe^t)$$

$$= pq(1-qe^t)^{-2}(e^t)$$

$$\frac{d^2}{dt^2}[M_X(t)] = pq\,(1-qe^t)^{-2}(e^t) + (e^t)(-2)(1-qe^t)^{-3}(-qe^t)$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = pq(1-q)^{-2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$E(X)^2 = \frac{d^2}{dt^2}[M_X(t)]_{t=0}$$

$$= pq[(p)^{-2} + 2q\,p^{-3}]$$

$$= pq\left[\frac{1}{p^2} + \frac{2q}{p^3}\right]$$

$$= \frac{q}{p} + \frac{(pq)\, 2q}{p^3}$$

$$= \frac{q}{p} + \frac{2q^2}{p^2} = E(x^2)$$

$$Var(X) = E(x^2) - [E(x)]^2$$

$$= \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2}$$

$$= \frac{q}{p} + \frac{q^2}{p^2} = \frac{pq + q^2}{p^2}$$

$$= \frac{q\,(p+q)}{p^2}$$

$$Var(X) = \frac{q}{p^2}.$$

Memory Loss property of GD:

If x has a probability geometric distribution than for any 2 integers m and n.

$$P\left[\frac{X > m+n}{X > m}\right] = P\,[X > n]$$

Proof:-

$$P\left[\frac{X > m+n}{X > m}\right] = \frac{P\,[X > m+n \;\cap X > m]}{P\,[X > m]}$$

$$\left[\because P\left(\frac{A}{B}\right) = \frac{P\,(A \cap B)}{P\,(B)}\right]$$

$$= \frac{P[X > m+n]}{P[X > m]} \rightarrow (1).$$

$$P(X=x) = q^{x-1} p. \quad x = 1,2,3.$$

Now,

$$P(X>K) = \sum_{K+1}^{\infty} q^{x-1} p$$

$$= [q^K p + q^{K+1} p + q^{K+2} p + \dots]$$

$$= q^K p [1 + q + q^2 + q^3 + \dots]$$

$$= q^K p [1-q]^{-1}$$

$$= q^K$$

$$\therefore P[X>m] = q^m$$

$$P[X>m+n] = q^{m+n}$$

eqn (1) ⇒

$$P\left[\frac{X>m+n}{X>m}\right] = \frac{q^{m+n}}{q^m}$$

$$= q^n$$

$$= P[X>n].$$

problem based on Binomial, poisson &
Geometric distribution.

1) The mean of BD is 20 & Standard deviation is 4. Determine the parameter of the distribution

$$Mean = 20 \quad SD = 4.$$

$$Mean = np$$

$$S.D = \sqrt{Var(X)}$$

$$(S.D)^2 = Var(X) = npq$$

$$np = 20$$

$$npq = 16$$

$$\frac{npq}{np} = \frac{16}{20}$$

$$q = \frac{4}{5}$$

$$p = 1-q \quad [\because p+q=1]$$

$$= 1-\frac{4}{5} = \frac{1}{5}$$

$$p = \frac{1}{5}$$

$$np = 20$$

$$n\left(\frac{1}{5}\right) = 20$$

$$n = 100.$$

2) The Mean and Variance of BD is 4 and $\frac{4}{3}$. Find $P(X \geq 1) = ?$

$$np = 4$$

$$npq = \frac{4}{3}$$

$$\frac{npq}{np} = \frac{4/3}{4} = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}$$

$$q = \frac{1}{3}$$

$$p = 1 - q \Rightarrow p = 1 - \frac{1}{3} = \frac{2}{3}$$

$$p = \frac{2}{3}$$

$$np = 4$$

$$\Rightarrow n\left(\frac{2}{3}\right) = 4^{2}$$

$$\boxed{n = 6}$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - [P(X = 0)]$$

$$P(X = x) = nC_x p^x q^{n-x}$$

$[\because nC_0 = 1]$

$$= 1 - \left[6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6\right]$$

$$= 1 - \left(\frac{1}{3}\right)^6 = 0.996.$$

A large concinement of electric bulb manufacture 10% are defective. Random sample 20 is taken for inspection. find the probability

i) All are good bulbs

ii) Atmost there are 3 defective bulbs.

iii) Exactly there are 3 defective bulbs.

Sln

$P = 10\% = \frac{10}{100}$

$= 0.1$

$q = 1-P = 0.9$

$n = 20$

$P(X=x) = nCx\, p^x q^{n-x}$

i) P [all are good bulbs]

$= P[X=0]$

$= 20C0\,(0.1)^0\,(0.9)^{20}$

$= (1)(1)(0.9)^{20}$

$= 0.1215$

ii) $P(X \le 3)$

Atmost $\le$

Atleast $\ge$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$P(X=x) = 20Cx\,(0.1)^x\,(0.9)^{n-x}$

$= 20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)^1 (0.9)^{19} + 20C_2 (0.1)^2 (0.9)^{18} + 20C_3 (0.1)^3 (0.9)^{17}$

$= (1)(1)(0.9)^{20} + 20 (0.1)(0.9)^{19} + 190 (0.1)^2 (0.9)^{18} + 1140 (0.1)^3 (0.9)^{17}$

$= 0.8666.$

iii) P[exactly 3 defective]

$= P[X=3]$

$= 20C_3 (0.1)^3 (0.9)^{17}$

$= 0.19$

4] A Screw is produced 5% defective random sample Screw is taken 15. What is probability

i) Exactly 3 defective

ii) Not more than 3 defective

5] 6 dices are thrown 729 times how many times do you expect atleast 3 dice to show 5 or 6.

Sln

P = Probability of getting 5 or 6 in 1 die

$= \frac{2}{6} = \frac{1}{3}$

$$p = \frac{1}{3}$$

$$q = 1 - p$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$P(X = x)$

$$= 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

P[Atleast 3 dices show 5 or 6]

$= P(X \geq 3)$

$= P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$$= 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + 6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 + 6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0$$

$$= \left(\frac{1}{3}\right)^6 \left[8(6C_3) + 4(6C_4) + 2(6C_5) + 1\right]$$

$$= \frac{1}{729}[233]$$

729 times,

$$= 729 \times \frac{1}{729}[233]$$

$= 233.$

Problem based on Poisson distribution.

1] The no. of monthly breakdown of computer with a random variable with a mean 1.8. Find the probability that these computer with function for a month?

i) without the breakdown.

ii) with only one breakdown.

X = No. of monthly breakdown of the computer.

$\lambda = 1.8$ [mean]

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{\lfloor x}$$

$$= \frac{e^{-1.8} (1.8)^x}{\lfloor x}$$

1) P[without breakdown]

$= P[X=0]$

$$= \frac{e^{-1.8} (1.8)^0}{\lfloor 0} = e^{-1.8}$$

2) P[only one breakdown]

$= P[X=1]$

$$= \frac{e^{-(1.8)} (1.8)^1}{\lfloor 1}$$

$= (1.8) e^{-1.8}$

$= 0.1652 \times 1.8$

$= 0.2975$

Uniform distribution :-

$$f(x) = \frac{1}{b-a}, \quad a<x<b.$$

The m.g.f $M_X(t) = \int_a^b e^{tx} f(x)\,dx$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{bt}-e^{at}}{t}\right] = \frac{e^{bt}-e^{at}}{(b-a)t}$$

$b^2-a^2 = (b-a)(b+a)$

$$= \frac{\left[1+\frac{bt}{1!}+\frac{(bt)^2}{2!}+\cdots\right]-\left[1+\frac{at}{1!}+\frac{(at)^2}{2!}+\cdots\right]}{(b-a)t}$$

$b^3-a^3 = (b-a)(b^2+ab+a^2)$

$$= \frac{\frac{(b-a)t}{1!}+\frac{(b^2-a^2)t^2}{2!}+\frac{(b^3-a^3)t^3}{3!}+\cdots}{(b-a)t}$$

$$= 1+\frac{(b+a)t}{2!}+\frac{(b^2+ba+a^2)t^2}{3!}+\cdots$$

$$M_X(t) = 1+\left(\frac{b+a}{2}\right)\frac{t}{1!}+\left[\frac{b^2+ba+a^2}{3}\right]\frac{t^2}{2!}+\cdots$$

Mean $E(X) = \text{co-eff of } \frac{t}{1!} = \frac{b+a}{2}$

$E(X^2) = \text{coeff of } \frac{t^2}{2!} = \frac{b^2+ba+a^2}{3}$

$Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{b^2+ba+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \left[\frac{b^2+ba+a^2}{3}\right] - \left[\frac{a^2+b^2+2ab}{4}\right]$$

$$= \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{1}{12}(b-a)^2$$

Exponential distribution :-

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \ \lambda > 0$$

The mgf $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda} dx$$

$$\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty} \qquad = \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda}{\lambda-t} = \frac{1}{1-\frac{t}{\lambda}}$$

$$= \left[1-\frac{t}{\lambda}\right]^{-1}$$

$$1+\frac{t}{1!}\mu_1+\frac{t^2}{2!}\mu_2 = 1+\frac{t}{\lambda}+\left(\frac{t}{\lambda}\right)^2+\cdots$$

$$= 1+\frac{1}{\lambda}t+\frac{1}{\lambda}t^2+\cdots$$

$$= 1+\frac{1}{\lambda}\frac{t}{1!}+\frac{2}{\lambda^2}\frac{t^2}{2!}+\cdots$$

Mean $E(X) = \text{coeff } \frac{t}{1!} = \frac{1}{\lambda}$

$E(X^2) = \text{coeff } \frac{t^2}{2!} = \frac{2}{\lambda^2}$

$Var(X) = E[X^2] - [E(X^2)]^2$

$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Normal distribution:-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The m.g.f.

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow \sigma z = x-\mu \Rightarrow \sigma dz = dx$,

$x = \sigma z + \mu \quad x \to -\infty \Rightarrow z \to -\infty \,;\; x \to \infty \Rightarrow z \to \infty.$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)} e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma t z + \mu t - \frac{1}{2}z^2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2\sigma t z)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left[z^2 - 2\sigma t z + \sigma^2 t^2\right]}$$

$$e^{1/2\, \sigma^2 t^2}\, dz$$

$$= \frac{e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\, (z-\sigma t)^2}\, dz$$

Put $u = z - \sigma t$, $du = dz$

$z \to \infty \Rightarrow u \to \infty$, $z \to \infty \Rightarrow u \to \infty$

$$= e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}\, du$$

$$= e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} \frac{1}{2\pi} \left(\sqrt{2\pi}\right)$$

$$M_X(t) = e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} \qquad \left[\because \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}}\, du = \sqrt{2\pi}\right]$$

$$\frac{d}{dt}\left[M_X(t)\right] = e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} \left(\mu + t\sigma^2\right)$$

$$\frac{d^2}{dt^2}\left[M_X(t)\right] = e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} (\sigma^2) + (\mu + t\sigma^2) \left[e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} (\mu + \sigma^2)\right]$$

$$\text{Mean} = E(X) = \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0} = \mu$$

$$E(X^2) = \left[\frac{d^2}{dt^2}\left[M_X(t)\right]\right]_{t=0} = \sigma^2 + \mu^2$$

$$\text{Var}(X) = E(X^2) - \left[E(X)\right]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

Memory less Property of exponential distribution

8m
(+)

If X is exponentially distributed then

$P(X > s + t / X > s) = P(X > t)$ for any $s, t \geq 0$.

Proof :-

$$P(X > k) = \int_k^{\infty} \lambda e^{-\lambda x} dx.$$

$$P(X > k) = \lambda \left[\frac{-e^{-\lambda x}}{\lambda} \right]_k^{\infty}$$

$$= e^{-\lambda k} \longrightarrow (1)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P[X > s + t / (X > s)] = \frac{P(X > s + t \cap X > s)}{P(X > s)}$$

$$= \frac{P(X > s + t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \quad \text{by (1)}$$

$$= e^{-\lambda t} = P(X > t)$$

Hence,

$$P(X > s + t / X > s) = P(X > t)$$

The converse of this result is also true.

1) A car hire firm as two cars. the no. of demands for as car on each day is distributed as poisson variant with mean 0.5 Find probability,

i) No car is used ii) some demand is refused

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{\lfloor x}$$

$\lambda = 0.5$

$$= \frac{e^{-0.5} (0.5)^{x}}{\lfloor x}$$

i) P [No car is used] = P [X=0]

$$= \frac{e^{-0.5} (0.5)^{0}}{\lfloor 0} = e^{-0.5} = 0.60$$

ii) P [some demand is refused]

$= P[X>2]$

$= 1 - P[X \leq 2]$

$= 1 - [P(X=0) + P[X=1] + P[X=2]]$

$$= 1 - \left[\frac{e^{-0.5}(0.5)^{0}}{\lfloor 0} + \frac{e^{-0.5}(0.5)^{1}}{\lfloor 1} + \frac{e^{-0.5}(0.5)^{2}}{\lfloor 2}\right]$$

$$= 1 - e^{-0.5}\left[1 + (0.5) + \frac{(0.5)^2}{2}\right]$$

$= 0.0145.$

2) The mgf the random variable is given by $M_X(t) = e^{3(e^t - 1)}$ find $P(X=1)$.

<u>Sln</u>

$M_X(t) = e^{\lambda(e^t - 1)}$

$\lambda = 3.$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$P(X=1) = \frac{e^{-3}(3)^1}{\lfloor 1} = 3e^{-3} = 0.149.$$

3) Write down the probability fn of P.D which approx. equal to B(100, 0.02)

$B(100, 0.02)$

$n = 100, p = 0.02$

$\lambda = np = (100)(0.02)$

$\boxed{\lambda = 2}$

PMF $P(X=x) = \frac{e^{-2}(2)^x}{\lfloor x}$

4) If X is a Poisson variant such that $P[X=1] = 3/10$ $P(X=2) = \frac{1}{5}$ find $(P(X=0))$ and $P(X=3)$.

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \frac{e^{-\lambda}\lambda}{\lfloor 1} = \frac{3}{10} \rightarrow (1)$$

$$= \frac{e^{-\lambda}\lambda^2}{\lfloor 2} = \frac{1}{5} \rightarrow (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{\frac{e^{-\lambda}\lambda^2}{\lfloor 2}}{\frac{e^{-\lambda}\lambda}{\lfloor 1}} = \frac{1/5}{3/10}$$

$$\frac{\lambda}{2} = \frac{1}{5} \times \frac{10}{3}$$

$$\boxed{\lambda = \frac{4}{3}}$$

$$P(X=0) = \frac{e^{-4/3}(4/3)^0}{\lfloor 0}$$

$$P(X=3) = \frac{e^{-4/3}(4/3)^3}{\lfloor 3}$$

5) A Manufacture of pins known as 2% of product are defective. If he sells pins in 100 Box & guaranty not more than 4 pins are defective. What is the probability that a box will fails to meet the guaranty quantity

$n=100 \quad p = 2\% = \frac{2}{100} = 0.02$

$\lambda = np = (100)(0.02)$

$\boxed{\lambda = 2}$

P [that box will fails to meet guaranty Quality]

$= P[X>4]$

$= 1 - P[X \leq 4]$

$= 1 - [P[X=0] + P[X=1] + P[X=2] + P[X=3] + P[X=4]$

$= 1 - \left[\frac{e^{-2}(2)^0}{\lfloor 0} + \frac{e^{-2}(2)^1}{\lfloor 1} + \frac{e^{-2}(2)^2}{\lfloor 2} + \frac{e^{-2}(2)^3}{\lfloor 3} + \right.$

$$\frac{e^{-2}(2)^4}{\lfloor 4}$$

0.55

$$= 1 - e^{-2}\left[1 + 2 + \frac{(2)^2}{2} + \frac{(2)^3}{3} + \frac{(2)^4}{4}\right]$$

$$= 1 - e^{-2}[1+2+2+1.33+0.66]$$

$$= 1 - 0.945 = 0.055$$

6) Messages arrives in poisson manner

H/W. find the probability i) exactly 2 msg arrives in 1hr. ii) No msg arrives in 1 hr iii) atleast 3 msg arrives in 1hr.

7 A problem based on G.D. If the probability an application will pass the road test on any given trail is 0.8. What is the probability. Find when finally pass the test. i) on the 4th trail. ii) In fewer than 4th trail [less than]

Sln

$$P(X=x) = q^{x-1} p$$

$p = 0.8 \quad x = 1,2,3$

$q = 1-p = 1-0.8 = 0.2$

i) $P(X=4) = (0.2)^3 \cdot 0.8.$

$= 0.0064$

ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= (0.2)^0(0.8) + (0.2)^1(0.8) + (0.2)^2(0.8)$

$= 0.992$

The Probability that target is destroy on one shot is 0.5. What is the probability that it would be destroyed on 6th trial.

$$p = 0.5$$

$$q = 1-p$$

$$= 1-0.5 = 0.5$$

$$P[X=6] = (0.5)^5 (0.5)$$

$$= 0.0156.$$

If X is a Geometric variant taking the value $x = 1, 2, \dots \infty$ find P [X is odd]

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, \dots \infty$$

$$P[X=\text{odd}] = P[x = 1, 3, 5, 7, \dots]$$

$$= P[X=1] + P[X=3] + P[X=5] + \dots$$

$$= p + q^2 p + q^4 p + \dots$$

$$= p[1 + q^2 + q^4 + \dots]$$

$$= p[1 + q^2 + (q^2)^2 + \dots]$$

$$[\because (1-x)^{-1} = 1 + x + x^2 + \dots]$$

$$= p[1-q^2]^{-1}$$

$$= \frac{p}{(1-q^2)}$$

$$= \frac{p}{(1+q)(1-q)}$$

$$= \frac{1}{1+q} \quad [\because p = 1-q]$$

Uniform Distribution.

X is a uniformly distribution with mean is 1 variance $= \frac{4}{3}$. find $P(X<0)$

Soln

Mean = 1 Variance $= \frac{4}{3}$

$$\text{Mean} = \frac{b+a}{2} = 1$$

$$\text{Variance} = \frac{(b-a)^2}{12} = \frac{4}{3}$$

$$b+a = 2$$

$$(b-a)^2 = 16$$

$$b-a = 4.$$

$$\Rightarrow \begin{array}{r} b+a = 2 \\ b-a = 4 \\ \hline 2b = 6 \end{array}$$

$$\boxed{b=3}$$

$$\Rightarrow b+a = +2$$

$$a = +2-3$$

$$\boxed{a=-1}$$

$$f(x) = \frac{1}{b-a} \quad a<x<b$$

$= \frac{1}{4} \quad -1 < x < 3$

$P[X<0] = \int_{-1}^{0} f(x)dx$

$= \int_{-1}^{0} \frac{1}{4} dx = \frac{1}{4}[x]_{-1}^{0}$

$= \frac{1}{4}[0+1] = \frac{1}{4}.$

Note:-

The PDF of a uniform variate $(-a,a)$ is given by. $f(x) = \frac{1}{2a} \quad -a<x<a$

$\quad 0 \quad$ O.W

The random variable X uniform distribution over $(-3,3)$. Find i) $P[X<2]$ ii) $P[|X|<2]$ iii) $P[|x-2|<2]$ 4) Find k where $P[X>k] = \frac{1}{3}$

$f(x) = \frac{1}{6}, \quad -3<x<3.$

i) $P(x<2) = \int_{-3}^{2} \frac{1}{x} dx$

$= \frac{1}{6}[x]_{-3}^{2}$

$= \frac{1}{6}[2+3] = \frac{5}{6}.$

ii) $P[|X|<2]$

$= P[-2<x<2]$

$= \int_{-2}^{2} \frac{1}{6} dx$

$$= \frac{1}{6}[x]_{-2}^{2}$$

$$= \frac{1}{6}[2+2] = \frac{4}{6} = \frac{2}{3}$$

3) $P[|x-2|<2]$

$$\Rightarrow P[-2<x-2<2]$$

$$= P[0<x<4]$$

$$= \int_0^3 \frac{1}{6}\,dx \Rightarrow \frac{1}{6}[x]_0^3$$

$$= \frac{1}{6}(3) \Rightarrow \frac{3}{6} = \frac{1}{2}$$

iv) $P[X>K] = \frac{1}{3}$

$$= \int_K^3 \frac{1}{6}\,dx = \frac{1}{3}$$

$$= \frac{1}{6}[x]_K^3 = \frac{1}{3}$$

$$= \frac{1}{\cancel{6}_2}[3-K] = \frac{1}{\cancel{3}}$$

$$= 3-K = 2$$

$$\boxed{K=1}$$

A random variable Y is defined $\cos\pi x$ where X is a uniform probability density fn. $(-\frac{1}{2}, \frac{1}{2})$. Find the mean & Std. deviation

$$f(x) = \frac{1}{1} = 1, \quad -\frac{1}{2}<x<\frac{1}{2}$$

Mean $E(x) = \int_{-\infty}^{\infty} y\, f(x)\,dx.$

$$= \int_{-1/2}^{1/2} \cos \pi x \,(1)\, dx$$

$$= \left[\frac{\sin \pi x}{\pi} \right]_{-1/2}^{1/2}$$

$$= \left[\frac{\sin \pi/2}{\pi} - \frac{\sin(-\pi/2)}{\pi} \right]$$

$$= \left[2 \frac{\sin \pi/2}{\pi} \right]$$

$$= \frac{2}{\pi} = 0.636$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f(x)\, dx$$

$$= \int_{-1/2}^{1/2} (\cos \pi x)^2 (1)\, dx$$

$$= \int_{-1/2}^{1/2} (\cos^2 \pi x)\, dx$$

$$\left[\begin{array}{l} \cos^2\theta = \frac{1+\cos 2\theta}{2} \\ \sin^2\theta = \frac{1-\cos 2\theta}{2} \end{array} \right]$$

$$= \int_{-1/2}^{1/2} \left[\frac{1+\cos^2(\pi x)}{2} \right] dx$$

$$= \frac{1}{2} \left[x \right]_{-1/2}^{1/2} + \frac{1}{2} \left[\frac{\sin 2\pi x}{2\pi} \right]_{-1/2}^{1/2}$$

$$= \frac{1}{2} \left[\frac{1}{2} + \frac{1}{2} \right] = 1/2.$$

$Var(y) = E(y^2) - [E(y)]^2$

$= \frac{1}{2} - (0.636)^2$

$S.D = \sqrt{Var(x)} = 0.096.$

Buses arrives at a specified stop at 15 mins interval starting at 7.am. that buses arrives 7, 7:15, 7:30, 7:45, ... If a passenger at the specific stop at random time is uniformly distributed b/w 7 & 7:30am. Find the probability range

i) less than 5 mins for a bus

ii) atleast 12 mins for a bus.

Sol:

Let x be uniformly distributed passenger arrives b/w 7 & 7:30 am.

$X \sim U(0, 30)$

$f(x) = \frac{1}{30-0} = \frac{1}{30}, \; 0<x<30.$

i) Passenger waits less than 5 mins he arrives b/w 7:10 to 7:15 & 7:25 to 7:30

$= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{30}[x]_{10}^{15} + \frac{1}{30}[x]_{25}^{30}$

$= \frac{1}{30}[5] + \frac{1}{30}[5] \Rightarrow \frac{2}{6} \Rightarrow \frac{1}{3}$

ii) Passenger waits atleast 12 mins

7-7:03 and 7:15 to 7:18

$$= \int_0^3 f(x)dx + \int_{15}^{18} f(x)\,dx$$

$$= \int_0^3 \frac{1}{30}dx + \int_{15}^{18} \frac{1}{30}dx$$

$$= \frac{1}{30}[3] + \frac{1}{30}[3]$$

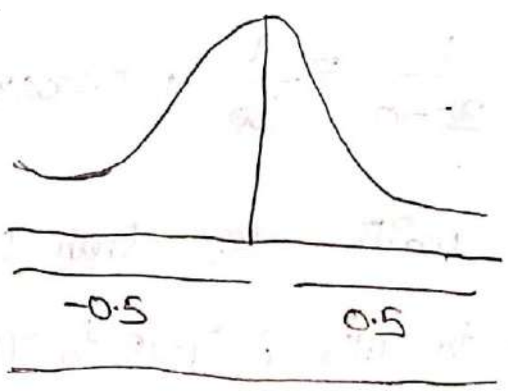
$$= \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5}.$$

Normal distribution:-

Let x be a continuous random variable its PDF is given by the

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

with mean μ, variance σ^2



An electric firm Manufacturer light bulbs that have a life before burn out. i.e., a normally distributed. with mean = 800hrs.

& std. deviation 40 hrs. find the probability

i) The bulbs burns more than 834 hrs.

ii) Bulbs burns b/w 778 & 834 hrs.

<u>Sln</u>

Let x random variable denoting the life time of bulbs given $\mu = 800$ hrs & $\sigma = 40$ hrs.

Normal variation,

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{x - 800}{40}$$

i) P [bulbs more than 834 hrs] = P[x > 834]

When x = 834

$$z = \frac{834 - 800}{40} = 0.85$$

$= P(z > 0.85)$

$= 1 - P(z < 0.85)$

$= 1 - 0.3023$ [normal tables]

$= 0.6977$.

ii) P [bulbs burns b/w 778 & 834] = P[778 < x < 834]

$$x = 778 \Rightarrow z = \frac{778 - 800}{40} = 0.55$$

$$x = 834 \Rightarrow z = \frac{834 - 800}{40} = 0.85$$

$P = (-0.55 < z < 0.85)$

$= P(-0.55 < Z < 0) + P[0 < Z < 0.85]$

$= P$ ~~0<Z<0.05~~ $[0 < Z < 0.55] +$

$P[0 < Z < 0.85]$

$= 0.2088 + 0.3023$

$= 0.511$

$P(Z > 0.85) = 0.5 - P(Z > 0.85)$

$= 0.1977$

A manufacturer produce airmail envelope whose wait is normal with mean ($\mu = 1.950$ gm), Std. deviation $\sigma = 0.025$ gms the envelope are stored in lot of 1000. How many envelope in a lot may be any of the two kgms

Let x be the random Variable which denoted the weight of an envelope

$\mu = 1.950$ gms

$\sigma = 0.025$ gm

Normal Variate $= \frac{x - \mu}{\sigma}$

When $x = 2$, $= \frac{x - 1.950}{0.025} = 2$

P [The envelope heavier than 2] = $P[x > 2]$

$= P[z>2]$

$= 0.5 - P[0<z<2]$

$= 0.5 - 0.4772$

$= 0.0228$.

Out of 1000, The envelope heavier than 2 gms

$= 1000 \times 0.0228$

$= 23$ (approx.)

The peak temperature has measure in degrees fahrenhit at a particular way is the Gaussian (85, 10). What is the probability of the i) greater than 100

ii) less than 60

iii) $P[70 \le T \le 100]$

Let T be the normal variable

$\mu = 85$

$\sigma = 10$

Normal variate $= \frac{T-\mu}{\sigma}$

$= \frac{X-85}{10} = \frac{100-85}{10}$

$= 1.5$

i) $= P(z>1.5)$

$= 0.5 - P(0>z>1.5)$

$= 0.5 - (0.4433)$

$= 0.066.$

ii) $P(T < 60)$

When $T = 60$

$$= \frac{60-85}{10} = -2.5$$

$P(T < 60) = P(Z < -2.5)$

$= 0.5 - P(0 < Z < 2.5)$

$= 0.5 - 0.4938$

$= 0.0062.$

iii) $P[70 \leq T \leq 100]$

When $T = 70$, $Z = \frac{70-85}{10}$

$= \frac{-15}{10} = -1.5$

$P(70 \leq T \leq 100) = P(-1.5 \leq Z \leq 1.5)$

$= P(-1.5 \leq Z \leq 0) + P(0 \leq Z \leq 1.5)$

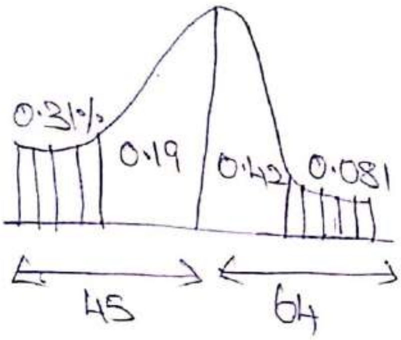
$= 0.4332 + 0.4332$

$= 0.8664.$

In a Normal distribution 31% of a items are under 45 and 8% over 64. Find the mean & std. deviation.

$31\% \rightarrow 45$

$81\% \rightarrow 64$



Let mean $= \mu$

S.D $= \sigma$ in N.D

The area lying to the ~~length~~ left

$0.5 - 0.31 = 0.19.$

At $x = 45$,

$$\frac{45-\mu}{\sigma} = -0.5$$

$\Rightarrow 45 - \mu = (-0.5)\sigma$

$\Rightarrow \mu - (0.5)\sigma = 45 \longrightarrow (1)$

The area lying to the right

$0.5 - 0.08 = 0.42$

At $x = 64$,

$$\frac{64-\mu}{\sigma} = 1.4$$

$\Rightarrow 64 - \mu = (1.4)\sigma.$

$\Rightarrow \mu + (1.4)\sigma = 64 \longrightarrow (2)$

Solving (1) & (2)

$$\mu - (0.5)\sigma = 45$$
$$\mu + (1.4)\sigma = 64$$
$$(-)\quad(-)\quad\quad(-)$$
$$-1.9\sigma = -19$$

$$\sigma = \frac{19}{1.9} = 10$$

Put $\sigma = 10$ in (2)

$\mu + (1.4)(10) = 64$

$\mu + 14 = 64$

$\mu = 64 - 14$

$\boxed{\mu = 50}$

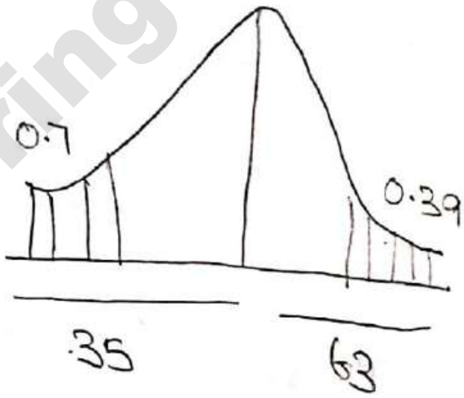
Exactly 7% of item under 35. 39% under 63. What are the Mean & S.D of the distribution.

$7\% \rightarrow 35$

$39\% = 63$

mean $= \mu$

S.D $= \sigma$



(X) The time in hours required to repair a machine is Exponential distribution with $\lambda = \frac{1}{2}$. i) What is the probability the repair time exceed 2 hrs.

ii) What is the conditional probabilities that a repair takes atleast 11 hrs that is duration exceed 8 hrs.

atleast 11 hrs given that its duration exceed 8 hr

X - repair time

$f(x) = \lambda e^{-\lambda x}$

$\lambda = 1/2$

$f(x) = 1/2 \, e^{-x/2}$

$$P[X>2] = \int_2^{\infty} \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2}\int_2^{\infty} e^{-x/2} dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^{\infty} \quad \left[e^{-\infty} - e^{-2/2}\right]$$

$$= -\left[e^{-x/2}\right]_2^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1}\right]$$

$$= e^{-1} \quad [\because e^{-\infty} = 0]$$

$$= 0.3679.$$

ii) $P[X>11 / x>8]$ $\quad \because P[x>s+t / x>s] = P[x>t]$

$$= P[X>3]$$

$$= \int_3^{\infty} \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_3^{\infty}$$

$$= -\left[0 - e^{-3/2}\right] = e^{-3/2} = 0.0248.$$

$\therefore P$

The mileage car owners certain kind of radial tire is a random variable having an ED with mean 4000 km. Find the Probability one of the tire will last.

i) Find the atleast 2000 km.
ii) Atmost 3000 km.

Mean = 4000 km

$\frac{1}{\lambda} = 4000 \Rightarrow \lambda = \frac{1}{4000}$

$f(x) = \lambda e^{-\lambda x}$

$x \geq 0, \lambda > 0.$

$\Rightarrow f(x) = \frac{1}{4000} e^{-\frac{x}{4000}}$

i) P[Atleast 2000] (or) $P[X > 2000]$

$= \int_{2000}^{\infty} \frac{1}{4000} e^{-\frac{x}{4000}} dx$

$= \frac{1}{4000} \left[\frac{e^{-\frac{x}{4000}}}{-1/4000} \right]_{2000}^{\infty}$

$= -\left[e^{-\infty} - e^{-1/2}\right] = e^{-1/2}$

$= 0.1839$

ii) P[Atmost 3000] or $P[X < 3000]$

$= \int_{0}^{3000} \frac{1}{4000} e^{-\frac{x}{4000}} dx$

$= \frac{1}{4000} \left[\frac{e^{-\frac{x}{4000}}}{-1/4000} \right]_{0}^{3000}$

$= -\left[e^{(-3/4)} - 1\right]$

$= 0.5276$

UNIT-2

)/1/19.

Unit-2

Two dimensional Random Variable

Marginal probability Mass fn:- of x :-

$$P_X(x_i) = P(X = x_i) = P_i$$

Marginal Probability Mass fn of y :-

$$P_Y(y_i) = P(Y = y_i) = P_{.J}$$

Conditional probability :-

The Conditional probability fn of x given $y = y_J$ is given formula.

$$P[X = x_i / Y = y_i] = \frac{P[X = x_i \cap Y = y_J]}{P[Y = y_J]}$$

Conditional probability fn of y is given $X = x_i$ is given formula.

$$P[Y = y_J / X = x_i] = \frac{P[X = x_i \cap Y = y_i]}{P[X = x_i]}$$

The two dimensional random variable fn of x & y are said to be independent $P[X = x_i, Y = y_i] = P[X = x_i] \times P[Y = y_i]$

[or]

$$P_{iJ} = P_i \times P_J$$

1) For the following table for bivariate distribution of (x,y) find i) $P(X \le 1)$ ii) $P(Y \le 3)$ iii) $P(X \le 1, Y \le 3)$ iv) $P(X \le 1 / Y \le 3)$ v) $P(Y \le 3 / X \le 1)$ vi) $P(X+Y \le 4)$

x\y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0.	$\frac{2}{64}$

$P_X(x)$

$P(X=0) = \frac{8}{24} = \frac{1}{4}$

$P(X=1) = \frac{10}{16} = \frac{5}{8}$

$P(X=2) = \frac{8}{64} = \frac{1}{8}$

$P_Y(y) = \frac{3}{32} \quad \frac{3}{32} \quad \frac{11}{64} \quad \frac{13}{64} \quad \frac{6}{32} \quad \frac{16}{24} \quad \frac{64}{64}$

$P(Y=1) \quad P(Y=2) \quad P(Y=3) \quad P(Y=4) \quad P(Y=5) \quad P(Y=6)$

i) $P(X \le 1) = P[X=0] + P[X=1]$

$= \frac{8}{32} + \frac{10}{16} = \frac{8+20}{32} = \frac{28}{32}$

ii) $P(Y \le 3) = P[Y=1] + P[Y=2] + P[Y=3]$

$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$

$= \frac{6+6+11}{64} = \frac{23}{64}$

3) $P(X \le 1, Y \le 3)$

$= P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)$

$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$= \frac{1+2+2+4}{32} = \frac{9}{32}$

4) $P[X \le 1 / Y \le 3]$

$= \frac{P[X \le 1, Y \le 3]}{P[Y \le 3]}$ $\left[\because P\left(\frac{A}{B}\right) = \frac{P[A \cap B]}{P[B]} \right]$

$= \frac{9/32}{23/64} = \frac{18}{23}$

5) $P[Y \le 3 / X \le 1]$

$= \frac{P[X \le 1, Y \le 3]}{P[X \le 1]}$

$= \frac{9/32}{28/32} = \frac{9}{28}$

6) $P(X + Y \le 4)$

$= P(0,1) + P(0,2) + P(0,3) + P(0,4) + P(1,1) + P(1,2) + P(1,3) + P(2,1) + P(2,2)$

$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$

$= \frac{13}{32}$.

The joint probability mass fn. of X, Y is given by P(X,Y) = k(2x+3y) where x = 0,1,2 & y = 1,2,3. Find all the marginal & conditional probability distribution of x, y & also find probability distribution of x+y. and P(x+y > 3)

Sol

$P(X,Y) = k(2x+3y)$

y \ x	0	1	2	Total
1	3k	5k	7k	15k
2	6k	8k	10k	24k
3	9k	11k	13k	33k
Total	18k	24k	30k	72k.

Total probability = 1

72k = 1

$k = \frac{1}{72}$

New table

X \ Y	0	1	2
1	3/72	5/72	7/72
2	6/72	8/72	10/72
3	9/72	11/72	13/72

Marginal distribution for fun of x

$P(X=0) = \frac{15}{72}$

$P(X=1) = 24/72$

$P(X=2) = \frac{30}{72}$

Marginal distribution fun. of y

$P(Y=1) = \frac{15}{72}$

$P(Y=2) = 24/72$

$P(Y=3) = 33/72$

conditional distribution function x given y.

$P[X=0/Y=1] = \frac{1}{5}$

$P\left(\frac{X=0}{Y=1}\right) = P\left[\frac{X=0, Y=1}{P\ Y=1}\right]$

$= \frac{3/72}{15/72}$

$= \frac{3}{15} = \frac{1}{5}$

$P[X=1/Y=1] = \frac{1}{8}$

$P[X=2/Y=1] = \frac{7}{15}$

$P[X=0/Y=2] = \frac{6}{24}$

$P[X=1/Y=2] = \frac{8}{24}$

$P[X=2/Y=2] = \frac{10}{24}$

$P[X=0/Y=3] = \frac{9}{33}$

$P[X=1/Y=3] = 11/33$

$P[X=2/Y=3] = 13/33$

Conditional distribution function of Y given X

$P(Y=1|X=0) = \frac{3}{18}$

$P(Y=2|X=0) = \frac{6}{18}$

$P(Y=3|X=0) = \frac{9}{18}$

$P(Y=1|X=1) = \frac{5}{24}$

$P(Y=2|X=1) = \frac{8}{24}$

$P(Y=3|X=1) = \frac{11}{24}$

$P(Y=1|X=2) = \frac{7}{30}$

$P(Y=2|X=2) = \frac{10}{30}$

$P(Y=3|X=2) = \frac{13}{30}$

(iii) probability distribution of $x+y$.

$x+y$		P
1		
$P(0,1)$	$\rightarrow$	$\frac{3}{72}$
2		
$P(1,1)+P(0,2)$	$\rightarrow$	$\frac{5}{72}+\frac{6}{72}=\frac{11}{72}$
3		
$P(0,3)+P(1,2)+P(2,1)$	$\rightarrow$	$\frac{9}{72}+\frac{8}{72}+\frac{7}{72}=\frac{24}{72}$
4		
$P(1,3)+P(2,2)$	$\rightarrow$	$\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$
5		
$P(2,3)$	$\rightarrow$	$\frac{13}{72}$
Total		1

$P(x+y \geq 3)$

$= P(x+y=4) + P(x+y=5) + P$

$= \frac{21}{72} + \frac{13}{72} = \frac{34}{72}$

Let x & y be a 2 random variable having a joint probability fn $f(x,y) = k(x+2y)$ where x & y can assume only integer value find the marginal & conditional distribution of x given y=1.

x = 0,1,2 & y = 0,1,2.

x / y	0	1	2	
0	0k	k	2k	=3k
1	2k	3k	4k	=9k
2	4k	5k	6k	=15k
Total	6k	9k	12k	27k

$k = \frac{1}{27}$

Marginal distribution of x.

$P(X=0) = \frac{6}{27}$

$P(X=1) = \frac{9}{27}$

$P(X=2) = \frac{12}{27}$

Marginal distribution y

$P(y=0) = \frac{3}{27}$

$P(y=1) = \frac{9}{27}$

$P(y=2) = 15/27$

Conditional distribution x ~~and~~ given y.

$P[x=0 / y=0] =$

$P[x=1 / y=0] =$

$P[x=2 / y=0] =$

$P[x=0 / y=1] =$

$P[x=1 / y=1] =$

$P[x=2 / y=1] =$

$P[x=0 / y=2] =$

$P[x=1 / y=2] =$

$P[x=2 / y=2] =$

Three balls are drawn at ~~random~~

conditional distribution y given x

$P[y=0 / x=0] =$

$P[y=1 / x=0] =$

$P[y=2 / x=0] =$

$P[y=0 / x=1] =$

$P[y=1 / x=1] =$

$P[y=2 / x=1] =$

$P(y=0 / x=2) =$

$P[y=1 / x=2] =$

$P[y=2 / x=2] =$

y \ x	0	1	2
0	0		
1	2/3		
2	4/3		

Three Balls are drawn at Random Without replacement from a box Containing 2 White, 3 red and 4 Black balls. If x denotes the no. of White balls drawn and y denote the no. of red balls drawn. Find the joint Probability distribution of (x, y).

$2W + 3R + 4B = 9$ balls

X - denote number of white balls

Y - denote number of Red balls

(W) X / Y (R)	0 WRB	1 WRB	2 WRB	3 WRB
0	$\frac{2C_0 \times 3C_0 \times 4C_3}{9C_3}$	$\frac{2C_0 \times 3C_1 \times 4C_2}{9C_3}$	$\frac{2C_0 \times 3C_2 \times 4C_4}{9C_3}$	$\frac{2C_0 \times 3C_3 \times 4C_0}{9C_3}$
1	$\frac{2C_1 \times 3C_0 \times 4C_2}{9C_3}$	$\frac{2C_1 \times 3C_1 \times 4C_1}{9C_3}$	$\frac{2C_1 \times 3C_2 \times 4C_0}{9C_3}$	0 only drawn 3 balls
2	$\frac{2C_2 \times 3C_0 \times 4C_1}{9C_3}$	$\frac{2C_2 \times 3C_1 \times 4C_0}{9C_3}$	0 only 3 balls are drawn.	0 only 3 balls drawn

Joint probability density function

1) $f(x,y) \geq 0$

2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$

Note:

1) $f(x,y) = \frac{\partial^2}{\partial x\, \partial y} F(x,y)$

2) If x & y independent
$f(x,y) = f(x) \cdot f(y)$

Marginal density fun. of x

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Marginal density fun. of y

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Conditional density function x given y or on y

$$f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f(y)}$$

C.D function y given x.

$$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$$

1) Joint Probability density fun. of x, y has

$$f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{o.w} \end{cases}$$

find the marginal & conditional D.F of x & y check x & y are independent or not.

JPDF $f(x,y)=\begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{o.w} \end{cases}$

Marginal density fun. of x

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\int_{x}^{1}(8xy)\,dy$$

$$=8x\left[\frac{y^2}{2}\right]_x^1$$

$$=4x\left[1-x^2\right]$$

Marginal density fun. of y

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

$$=\int_{0}^{y}(8xy)\,dx$$

$$=8y\left[\frac{x^2}{2}\right]_0^y$$

$$=4y\left[y^2-0\right]$$

$$=4y^3,\quad 0<y<1$$

Marginal density fun. x given y

$$f(x/y)=\frac{f(x,y)}{f(y)}$$

$$=\frac{8xy}{4y^3}$$

$$=\frac{2x}{y^2},\quad 0<x<y.$$

Conditional density function y on x.

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)}$$

$$f(y/x) = \frac{2y}{1-x^2}, \quad x<y<1$$

x and y are independent

To prove

$$f(x,y) = f(x)\,f(y)$$

$$f(x)f(y) = 4x(1-x^2)\cdot 4y^3$$

$$= 16xy^3(1-x^2)$$

$$\neq f(x,y) \quad \therefore x \,\&\, y \text{ are independent.}$$

J.P.D. fun of $f(x,y)$ are given by
$f(x,y) = kxy\,e^{-(x^2+y^2)}, x>0, y>0$, i) find the value of K P.T x, y are also independent.

~~f(x,y)~~ By formula,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

$$\Rightarrow \int_0^{\infty}\int_0^{\infty} kxy\,e^{-x^2} e^{-y^2}\,dx\,dy = 1.$$

$$\Rightarrow K\int_0^{\infty} xe^{-x^2}dx \int_0^{\infty} ye^{-y^2}dy = 1.$$

$[\because \int xe^{-x^2}dx.$

put $t = x^2$ put $x=0 \Rightarrow t=0$

$\frac{dt}{dx} = 2x$ put $x=\infty$, $t=\infty$

$$\frac{dt}{2} = x\,dx \Big]$$

$$\int_0^\infty x e^{-x^2} dx = \int_0^\infty e^{-t}\left(\frac{dt}{2}\right)$$

$e^{-0} = 1$

$e^{-\infty} = 0$

$$= \frac{1}{2}\left[-e^{-t}\right]_0^\infty = -e^{-\infty} + e^{-0} = \frac{1}{2}(1)$$

$$\Rightarrow K\left(\frac{1}{4}\right) = 1$$

$$\boxed{K = 4}$$

To prove $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Marginal D.f of X

$$= \int_0^\infty kxy\, e^{-(x^2+y^2)} dy$$

$$= 4x \int_0^\infty y\, e^{-x^2} e^{-y^2} dy$$

$$= 4x e^{-x^2} \int_0^\infty y\, e^{-y^2} dy$$

$$= 4x e^{-x^2}\left(\frac{1}{2}\right) = 2x e^{-x^2}$$

Marginal Density function of Y.

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^\infty 4xy\, e^{-x^2} e^{-y^2} dx$$

$$= 4y e^{-y^2} \int_0^\infty x e^{-x^2} dx$$

$$= 4y e^{-y^2}(1/2) = 2y e^{-y^2} = 1$$

$$f(x) \cdot f(y) = 2x e^{-x^2}\, 2y e^{-y^2}$$

$= 4xy\, e^{-(x^2+y^2)}$

$= f(x,y)$

x & y are independent

JPDF is $\begin{cases} cx(x-y), & 0<x<2, -x<y<x \\ 0 & o.w \end{cases}$

find the c value & x & y are independent

$f(x,y) = cx(x-y)$ [Pg: 2.28]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

A JPDF is given by $f(x,y) = \begin{cases} 1/8\,(6-x-y) & 0<x<2, 2<y<4 \\ 0 & o.w \end{cases}$

i) $P(X<1 \cap Y<3)$ 2) $P(X<1 / Y<3)$

3) $P(x+y<3)$

Sln

i) $P(X<1 \cap Y<3) = \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$= \frac{1}{8}\int_0^1 \left[6y - xy - y^2/2\right]_2^3 dx$

$= \frac{1}{8}\left[\int_0^1 \left(18 - 3x - \frac{9}{2}\right) - \left(12 - 2x - \frac{4}{2}\right)\right] dx$

$= \frac{1}{8}\left[\int_0^1 6 - x - \frac{5}{2}\right] dx$

$= \frac{1}{8}\left[6x - \frac{x^2}{2} - \frac{5}{2}(x)\right]_0^1$

$= \frac{1}{8}\left[6 - \frac{1}{2} - \frac{5}{2}\right]$

$$= \frac{1}{8}\left[\frac{12-1-5}{2}\right]$$

$$= \frac{6}{16} = \frac{3}{8}$$

ii) $P(X<1 / Y<3)$

$$= \frac{P[X<1 \cap Y<3]}{P[Y<3]}$$

$$P(Y<3) = \int_0^2 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^2 \left(6-x-\frac{5}{2}\right) dx$$

$$= \frac{1}{8}\left[6x-\frac{x^2}{2}-\frac{5}{2}x\right]_0^2$$

$$= \frac{1}{8}\left[12-\frac{4}{2}-\frac{5}{2}(2)\right]$$

$$= \frac{1}{8}[12-2-5]$$

$$= \frac{5}{8}$$

$$= \frac{3/8}{5/8} = \frac{3}{5}.$$

iii) $P(x+y<3) = \int_0^3 \int_2^{3-x} \frac{1}{8}(6-x-y)\,dy\,dx$

$x+y=3$

$y=3-x$

$x=3-y$

$$= \frac{1}{8}\int_0^2 \left[6y-xy-y^2/2\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^3 \left[6(3-x)-x(3-x)-\frac{(3-x)^2}{2}\right] -$$

$[12-2x-4/2]\,dx$

$$= \frac{1}{8}\left[\int_0^3 18-6x-3x+x^2-\frac{1}{2}\left[9+x^2-6x\right]\right.$$
$$\left.-12+2x+2\right]dx$$
$$= \frac{1}{8}\int_0^3 18-6x-3x+x^2-\frac{9}{2}-\frac{x^2}{2}+\frac{6x}{2}$$
$$-12+2x+2]\,dx$$
$$= \frac{1}{8}\int_0^3 \left[6-4x+\frac{x^2}{2}-\frac{9}{2}+2\right]dx$$
$$= \frac{1}{8}\quad 6-12+\frac{9}{2}-\frac{9}{2}+2-\left[6-\frac{9}{2}+2\right] \quad \frac{3}{24}$$

$f(x,y) = x^2+xy/3,\ 0\le x\le 1\quad 0\le y\le 2$

Find the marginal density fn and conditional density fn iii) $P[X>\frac{1}{2}]$ iv) $P[Y<X]$

Marginal density fn of X

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \left(x^2+\frac{xy}{3}\right)dy$$
$$= \left[x^2y+\frac{x}{3}\left(\frac{y^2}{2}\right)\right]_0^2$$
$$= \left[2x^2+\frac{x}{3}\left(\frac{4}{2}\right)\right]$$
$$= 2x^2+\frac{2x}{3}$$
$$f(x) = 2\left(x^2+\frac{x}{3}\right)$$

Marginal density function of y.

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^1 \left(x^2+xy/3\right)dx$$
$$= \left[\frac{x^3}{3}+\frac{y}{3}\left(\frac{x^2}{2}\right)\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{y}{3}\left(\frac{1}{2}\right)\right]$$

$$= \left[\frac{1}{3} + \frac{y}{6}\right]$$

$$= \frac{1}{3}\left[1 + \frac{y}{2}\right]$$

$$f(y) = \frac{1}{3}\left[\frac{y}{2} + 1\right]$$

Conditional density fn x on y

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{x^2 + xy/3}{\frac{1}{3}\left(\frac{y}{2}+1\right)}$$

Conditional density fn y on x

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{x^2 + xy/3}{2(x^2 + x/3)}$$

$$P\left[X > \frac{1}{2}\right] = \int_{1/2}^{1} \left[\int_0^2 f(x,y)\,dy\right] dx$$

$$= \int_{1/2}^{1} \left[\int_0^2 \left(x^2 + \frac{xy}{3}\right) dy\right] dx$$

$$= \int_{1/2}^{1} \left[x^2 y + \frac{x}{3}\left(\frac{y^2}{2}\right)\right]_0^2 dx$$

$$= \int_{1/2}^{1} \left[2x^2 + \frac{x}{3}\left(\frac{4}{2}\right)\right] dx = \left[\frac{2x^3}{3} + \frac{2}{3}\left(\frac{x^2}{2}\right)\right]_{1/2}^{1}$$

$$= \left[\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)\right] - \frac{2(1/8)}{3} + \frac{2}{3}\frac{(1/4)}{2}$$

$$= \left[\frac{2}{3} + \frac{1}{3} - \frac{1}{4}\left(\frac{1}{3}\right) + \frac{2}{3}\left(\frac{1}{4}\right) \times \frac{1}{2}\right]$$

$$= \frac{5}{6}$$

$$P[Y<x] = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \frac{7}{24}.$$

H/W $f(x,y) = xy^2 + \frac{x^2}{8}$ $0 \le x \le 2$, $0 \le y \le 1$

Find the marginal density fn and conditional density fn i) $P[X>1 / Y<\frac{1}{2}]$ ii) $P[Y<\frac{1}{2} / X>1]$ iii) $P[X<Y]$

Correlation

$$r = \frac{\sum xy - \frac{\sum x \sum y}{N}}{\sqrt{\sum x^2 - \frac{(\sum x)^2}{N}} \sqrt{\sum y^2 - \frac{(\sum y)^2}{N}}}$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy.$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy.$$

Marginal density fn. of x

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Marginal density fn. of y

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

Eg: 2.2.8. [Refer in book]

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

Marginal density fn of x

$$f(x) = \int_{-\infty}^{\infty} f(x,y)$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1$$

$$= x + \frac{1}{2}, \; 0<x<1$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^1$$

$$= y + \frac{1}{2}.$$

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x\left(x+\frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(Y) = \int_0^1 y f(y)\,dy = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1$$

$$= \frac{7}{12}$$

$$[E(X)]^2 = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^3 + \frac{x^2}{2}\right)dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{3+2}{12} = \frac{5}{12}.$$

$$E(Y^2) = \frac{5}{12}.$$

$V(X) = E(X^2) - E(X)^2$

$= \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$

$\sigma x = \frac{\sqrt{11}}{12} \quad \sigma y = \frac{\sqrt{11}}{12}$

$r_{xy} = \frac{cov(x,y)}{\sigma x \cdot \sigma y} = \frac{E(XY) - E(X)E(Y)}{\sigma x \cdot \sigma y}$

$E(XY) = \int_0^1 \int_0^1 (xy)(x+y)\,dx\,dy = \int_0^1 \int_0^1 (x^2y + xy^2)\,dx\,dy$

$= \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$

$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

$cov(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12} = \frac{1}{3} - \frac{49}{144} = \frac{48-49}{144} = \frac{-1}{144}$.

x & y are independent.

$r_{xy} = \frac{-1/144}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}} = \frac{-1}{11} = 0.0909\,(+ve)$

Regression.

1) The Regression eqn x on y

$x - \bar{x} = r \frac{\sigma x}{\sigma y}(y - \bar{y})$

2) Regression eqn y on x.

$y - \bar{y} = r \frac{\sigma y}{\sigma x}(x - \bar{x})$

3) The angle b/w two regression.

$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma y \sigma x}{\sigma y^2 + \sigma x^2}\right)$

Regression co-eff x on y

$$b_{xy} = r \frac{\sigma x}{\sigma y}$$

Regression co-eff y on x

$$b_{yx} = r \frac{\sigma y}{\sigma x}$$

$$r = \pm \sqrt{b_{xy}\, b_{yx}}$$

$$b_{xy} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma y^2 - \frac{(\Sigma y)^2}{N}}$$

$$b_{yx} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma x^2 - \frac{(\Sigma x)^2}{N}}$$

$$\sigma x = \sqrt{\frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2}$$

$$\sigma y = \sqrt{\frac{\Sigma y^2}{N} - \left(\frac{\Sigma y}{N}\right)^2} \quad (or)$$

$$b_{yx} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (x-\bar{x})^2}$$

$$b_{xy} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (y-\bar{y})^2}$$

2.3.1

Pg:2.85 From the following data, find i) the two regression eqn, ii) the coefficient of correlation b/w the marks in economics and statistics, iii) the most likely marks in Statistics when marks in economics are 30.

Marks in Economics x	25	28	35	32	31	36	29	38	34	32
Statistics y	43	46	49	41	36	32	31	30	33	39.

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	-32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	1	0	1	0
320	380	0	0	140	398	-93

$$\bar{x} = \frac{\Sigma x}{n} = \frac{320}{10} = 32$$

$$\bar{Y} = \frac{\Sigma y}{n} = \frac{380}{10} = 38$$

$$b_{yx} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (x-\bar{x})^2} = \frac{-93}{140} = -0.6643$$

$$b_{xy} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (y-\bar{y})^2} = \frac{-93}{398} = -0.2337.$$

i) Eqn of the line of regression of x on y is

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

$$x-32 = -0.2337(y-38)$$

$$x = -0.2337y + 40.8806.$$

Eqn of line of regression y on x is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y-38 = -0.6643(x-32)$$

$$y = -0.6643x + 59.2576$$

ii) Co-eff of correlation

$$r^2 = b_{yx} b_{xy} = (-0.6643)(-0.2337)$$

$$= 0.1552$$

$$r = \pm 0.394.$$

iii)

$$y = -0.6643 + 59.2575$$

$$x = 30 \Rightarrow y = 39.$$

2.3.2

Pg: 2.87 Two lines regression are $8x - 10y + 66 = 0$ — (A) $40x - 18y - 214 = 0$ → B. The variance of x is 9 find i) The mean values of x & y ii) correlation co-eff. b/w x & y.

Since both the lines of regression passes through the mean values $\bar{x}$ and $\bar{y}$, the pt $(\bar{x}, \bar{y})$ must satisfy the 2 given regression lines

$8\bar{x} - 10\bar{y} = -66 \rightarrow (1)$

$40\bar{x} - 18\bar{y} = 214 \rightarrow (2)$

$(1) \times 5 \quad 40\bar{x} - 50\bar{y} = -330 \rightarrow (3)$

$(2) \times 1 \quad 40\bar{x} - 18\bar{y} = 214 \rightarrow (4)$

$(3) - (4) \qquad -32\bar{y} = -544$

$\bar{y} = 17.$

eqn (1) we get,

$8\bar{x} - 10(17) = -66$

$\bar{x} = 13$

The mean values are given by $\bar{x} = 13$, $\bar{y} = 17$.

ii)

(A) $\Rightarrow 8x = 10y - 66$ $x = \frac{10}{8}y - \frac{66}{8}$ $b_{xy} = \frac{10}{8}$	B) $18y = 40x - 214$ $y = \frac{40}{18}x - \frac{214}{18}$ $b_{yx} = \frac{40}{18}$	$r^2 = b_{xy}\, b_{yx}$ $= \left(\frac{10}{8}\right)\left(\frac{40}{18}\right)$ $= 2.77$ $r = 1.66 \nless 1$
(A) $\Rightarrow 10y = 8x + 66$ $y = \frac{8}{10}x + \frac{66}{10}$ $b_{yx} = \frac{8}{10}$	B) $\Rightarrow 40x = 18y + 214$ $x = \frac{18}{40}y + \frac{214}{40}$ $b_{xy} = \frac{18}{40}$	$r^2 = b_{yx}\, b_{xy}$ $= \left(\frac{8}{10}\right)\left(\frac{18}{40}\right)$ $= 0.36$ $r = \pm 0.6$

Since both the regression coeff are positive r must be positive $r = 0.6$.

that RVs

how find the recursive eqn of the follow. data.

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
65	67	−3	−2	9	4	6
66	68	−2	−1	4	1	2
67	65	−1	−4	1	16	4
67	68	−1	−1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
544	552	0	6	36	44	24

$\bar{x} = 68, \ \bar{y} = 69$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{24}{36} = 0.666$$

$$b_{xy} = \frac{24}{44} = 0.5454$$

Regression Co-eff of x on y

$x-\bar{x} = b_{xy}(y-\bar{y})$

$x-68 = 0.66y(y-69)$

$x-68 = 0.66y - 45.954$

$x - 0.66y = 22.046$

Two regression lines are

$8x - 10y + 66 = 0$

$40x - 18y - 214 = 0$

Variance of x in a final mean value of x & y.

ii) correlation co-eff b/w x & y

$8x - 10y = +66$

$40x - 18y = 214$

-32

$① \times 5 \Rightarrow 40x - 50y = -330$

$② \times 1 \Rightarrow 40x - 18y = 214$

$-32y = -54$

$\bar{y} = 17$

$8\bar{x} - 10(17) = -66$

$8\bar{x} - 170 = 66$

$8\bar{x} = 104$

$\bar{x} = 13$

The mean value of $\bar{x} = 13$, $\bar{y} = 17$.

ii) $8x = 10y - 66$

$x = \frac{10}{8} y - \frac{66}{8}$

$b_{xy} = \frac{10}{8}$

$18y = 40x - 214$

$y = \frac{40}{18} x - \frac{214}{8}$

$b_{yx} = \frac{40}{18}$

$r^2 = b_{xy} - b_{yx}$

$= \left(\frac{10}{8}\right)\left(\frac{40}{18}\right) = 1.6641.$

iii) $10y = 8x + 66$

$y = \left(\frac{8}{10}\right)x + \frac{66}{10}$

$b_{yx} = \frac{8}{10}$

$40x = 18y + 214$

$x = \left(\frac{18}{40}\right)y + \frac{214}{40}$

$b_{xy} = \frac{18}{40}$

$$r^2 = b_{yx} - b_{xy}$$
$$= \left(\frac{8}{10}\right)\left(\frac{+8}{10}\right)$$
$$= \pm\ 0.06.$$

In regression eqn x & y $3y - 5x + 108 = 0$ mean value of y is 44 & variance of x $\frac{9}{6}$ var of (y). find the mean value of x & calculate co-efficient.

The given eqn passes through $(\bar{x}, \bar{y})$

$$3\bar{y} - 5\bar{x} + 108 = 0$$
$$-5\bar{x} = -3\bar{y} - 108$$
$$= -3(44) - 108$$
$$-5\bar{x} = -132 - 108$$
$$-5\bar{x} = -240$$
$$\bar{x} = \frac{240}{5} = 48$$
$$\therefore \bar{x} = 48.$$

$\therefore$ Mean value of $x = 48$.

given,

$$-5x = -3y - 108$$
$$x = \frac{3}{5}y + \frac{108}{5}$$

This is regression eqn x on y.

$b_{xy} = \frac{3}{5}$ — [regression co-eff x on y]

Given, $Var(X) = \frac{9}{16} Var(Y)$

$\Rightarrow \frac{\sigma x^2}{\sigma y^2} = \frac{9}{16}$

$\frac{\sigma x}{\sigma y} = \frac{3}{4}$ $\therefore Var(X) = \sigma x^2$

$\sqrt{Var(X)} = S.D\ f(x)$

$b_{xy} = r \frac{\sigma x}{\sigma y}$

$\frac{3}{5} = r\left(\frac{3}{4}\right)$

$r = \frac{4}{5} = 0.8$

The independent Random Variable X & Y have the variance 36, 16 find the correlation ~~regression~~ co-eff b/w $x+y$, $x-y$

formula Correlation:-

1) If change in 1 variable affect a change in another variable it said to be correlated

2) If X & Y are independent.

$E(XY) = E(X)E(Y)$

3) $Cov(X,Y) = E(XY) - E(X)E(Y)$

If X & Y are independent.

$= E(X)E(Y) - E(X)E(Y)$

$Cov(X,Y) = 0.$

Note:-

$Cov(ax, by) = ab\,cov(x, y)$

$Cov(x+a, y+b) = cov(x, y)$

$Cov(ax+b)(cy+d) = ac\ cov(x, y)$

$Var(x+y) = Var(x) + Var(y) + 2cov(x, y)$

$Var(x-y) = Var(x) + Var(y) - 2cov(x, y)$

Correlation coeff (r)

$$r = \frac{cov(x, y)}{\sigma x \cdot \sigma y} \text{ (or) } \frac{cov(x, y)}{\sqrt{Var(x)}\sqrt{Var(y)}}$$

$$Cov(x, y) = \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$$

$$\sigma x = \sqrt{\frac{1}{n}\Sigma x^2 - (\bar{x})^2}$$

$$\sigma y = \sqrt{\frac{1}{n}\Sigma y^2 - (\bar{y})^2}$$

$$\bar{x} = \frac{\Sigma x}{n} \cdot \bar{y} = \frac{\Sigma y}{n}$$

$Cov(x, y) = 0$

Let $U = x+y$

$V = x-y$

$$r = \frac{cov(U, V)}{\sigma U^2 \sigma V^2}$$

$cov(U, V) = E(UV) - E(U)E(V)$

$$= E[(X+Y)(X-Y)] - E[(X+Y)]\,E[X-Y]$$

$$= E[X^2-Y^2] - [E[X]+E[Y]]\,[E[X]-E[Y]]$$

$$= E(X^2) - E(Y^2) - [E(X)]^2 - E(X)E(Y) + E(X)E(Y) - [E(Y)]^2$$

$$= E(X^2) - E(Y^2) - [E(X)]^2 + [E(Y)]^2$$

$$= E(X^2) - [E(X)]^2 - \left(E(Y^2) - [E(Y)]^2\right)$$

$$= Var(X) - Var(Y)$$

$$= 36 - 16 = 20.$$

$$\sigma_U^2 = \sqrt{Var(U)}, \quad \sigma_V^2 = \sqrt{Var(V)}$$

$$Var(U) = Var(X+Y)$$

$$= Var(X) + Var(Y) + 2Cov(X,Y)$$

$$Var(V) = Cov(X-Y)$$

$$= Var(X) + Var(Y) - 2Cov(X,Y)$$

If X & Y are independent.

$$Cov(X-Y) = 0.$$

$$Var(U) = 36+16 = 52$$

$$Var(V) = 36+16 = 52$$

$$r = \frac{20}{\sqrt{52}\sqrt{52}} = \frac{20}{52} = 0.384.$$

Find the correlation coeff. follow. datas

65

$\bar{Y} = \frac{\Sigma Y}{n} = \frac{280}{7} = 40$

$\bar{X} = \frac{\Sigma X}{n} = \frac{410}{7} = 58.57 = 59$

X	Y	$u = x - \bar{x}$	$v = y - \bar{y}$	uv	u^2	v^2
55	35	-4	-5	20	16	25
56	38	-3	-2	6	9	4
58	37	-1	-3	3	1	9
59	39	0	-1	0	0	1
60	44	1	4	4	1	16
61	43	1	3	3	1	9
62	44	3	4	12	9	16

$$r = \frac{\text{cov}(U,V)}{\sigma U \cdot \sigma V} \text{ (or) } \frac{\text{cov}(U,V)}{\sqrt{\text{Var}\,U}\sqrt{\text{Var}\,V}}$$

$\bar{U} = \frac{\Sigma u}{n} = \frac{-3}{7} = -0.43$

$\bar{V} = \frac{\Sigma v}{n} = \frac{0}{7} = 0$

$$\text{Cov}(U,V) = \frac{\Sigma UV}{n} - \bar{U}\bar{V}$$

$$\sigma U = \sqrt{\frac{\Sigma U^2}{n} - (\bar{U})^2} = \sqrt{\frac{37}{7} - (-0.43)^2} = 2.2584$$

$$\sigma V = \sqrt{\frac{\Sigma V^2}{n} - (\bar{V})^2} = \sqrt{\frac{80}{7} - (0)^2} = 3.380$$

$$\text{Cov}(U,V) = \frac{48}{7} - 0 = 6.85$$

$$r = \frac{6.85}{(2.2584)(3.380)} = 0.897.$$

Find the Cartesian correlation Co-eff. b/w the follow. datas.

X	65	67	66	71	67	70	68	69
Y	67	68	68	70	64	67	72	70

X	Y	$u = x - \bar{x}$	$v = y - \bar{y}$	uv	u^2	v^2
65	67	-3	-1	3	9	1
67	68	-1	0	0	1	0
66	68	-2	0	0	4	0
71	70	3	2	6	9	4

67	64	-1	-4	4	1	16
70	67	2	-1	-2	4	1
68	72	0	4	0	0	16
69	70	1	2	2	1	4
					29	42

$\bar{x} = \frac{\Sigma x}{n} = \frac{543}{8} = 67.875 = 68$

$\bar{y} = \frac{\Sigma y}{n} = \frac{546}{8} = 68.25 = 68.$

$\gamma = \frac{cov(u,v)}{\sigma u \cdot \sigma v}$ (or) $\frac{cov(u,v)}{\sqrt{Var\, u}\sqrt{Var\, v}}$

$\bar{u} = \frac{\Sigma u}{n} = \frac{-1}{8} = -0.125$

$cov(u,v) = \frac{\Sigma uv}{n} - \bar{u}\bar{v}$

$\sigma u = \sqrt{\frac{\Sigma u^2}{n} - (\bar{u})^2} = \sqrt{\frac{29}{8} - (-0.125)^2}$

$\bar{v} = \frac{\Sigma v}{n} = \frac{2}{8} = 0.25$

$= \sqrt{3.625 - 0.0156} = \sqrt{3.6094}$

$= 1.8998$

$\sigma v = \sqrt{\frac{\Sigma v^2}{n} - (\bar{v})^2} = \sqrt{\frac{42}{8} - (0.25)^2}$

$= \sqrt{5.25 - 0.0625} = \sqrt{5.1875} = 2.2776.$

$cov(u,v) = \frac{13}{8} - 0.03125$

$= 1.625 - 0.03125 = 1.5937$

$\gamma = \frac{1.5937}{(1.8998)(2.2776)} = \frac{1.5937}{4.326}$

$\gamma = 0.3682.$

Transformation on random Variables :-

The JPdf,

$$f(u,v) = f(x,y)\,|J| \quad \text{where } J = \frac{\partial(x,y)}{\partial(u,v)}$$

(or)

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Pdf of 'u'

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv$$

Pdf of 'v'

$$f(v) = \int_{-\infty}^{\infty} f(u,v)\,du$$

Assume $v=y$

J.P.D.F of 2 dimensional random variable x and y is given by

$$f(x,y) = \begin{cases} 4xy\,e^{-(x^2+y^2)} & x\geq 0, y\geq 0 \\ 0 & \text{o.w} \end{cases}$$

$u = \sqrt{x^2+y^2}$

The JPDf.

$$f(u,v) = f(x,y)\,|J|$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Let us assume,

$u = \sqrt{x^2+y^2} \qquad v=y$

$u^2 = x^2+y^2 \qquad y=v.$

$$u^2-y^2=x^2$$

$$x^2=u^2-v^2$$

P.D. w.r.t to u and v,

$$2x\frac{\partial x}{\partial u}=2u$$

$$\frac{\partial x}{\partial u}=\frac{u}{x}$$

$$\Rightarrow 2x\frac{\partial x}{\partial v}=\frac{-v}{x}$$

$$\left.\begin{array}{l}\frac{\partial y}{\partial u}=0\\ \frac{\partial y}{\partial v}=1\end{array}\right\} \text{②}$$

$$\left.\begin{array}{l}\frac{\partial x}{\partial u}=\frac{u}{\sqrt{u^2-v^2}}\\ \frac{\partial x}{\partial v}=\frac{-v}{\sqrt{u^2-v^2}}\end{array}\right\} \text{①}.$$

$$J=\begin{vmatrix}\frac{u}{\sqrt{u^2-v^2}} & \frac{-v}{\sqrt{u^2-v^2}}\\ 0 & 1\end{vmatrix}$$

$$|J|=\frac{u}{\sqrt{u^2-v^2}}$$

$$f(x,y)=4xye^{-(x^2+y^2)}$$

$$=4(\sqrt{u^2-v^2})ve^{-u^2}$$

JPDF

$$f(u,v)=4(\sqrt{u^2-v^2})ve^{-v}\left[\frac{u}{\sqrt{u^2-v^2}}\right]$$

$$f(u,v)=4uve^{-v}$$

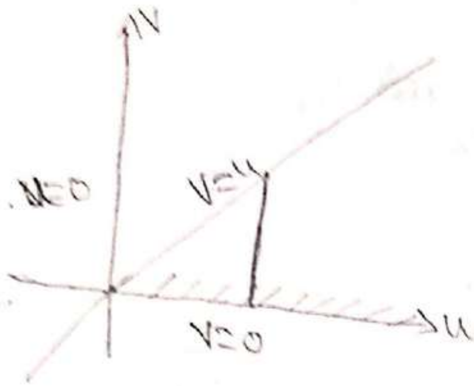
Range $x\geq 0,\ y\geq 0$

$\Rightarrow x^2\geq 0$ | $y\geq 0$

$\Rightarrow u^2-v^2\geq 0$ | $v\geq 0$ (or) $v>0$

$\Rightarrow u\geq v$ (or) | $v=0$

$u > v,\ u = v.$



The P.D.F is 'u',

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv$$

$$= \int_0^u 4uve^{-u^2}\,dv$$

$$= 4u\int_0^u ve^{-u^2}\,dv$$

$$= 4ue^{-u^2}\left[\frac{v^2}{2}\right]_0^u$$

$$= 4ue^{-u^2}\left[\frac{u^2}{2}\right] \Rightarrow 2u^3e^{-u^2}$$

(X) In the J.P.D.F of a 2 dimensions of the random variable is given by

$$f(x,y) = \begin{cases} x+y & 0 \le (x,y) \le 1 \\ 0 & \text{o.w} \end{cases}$$ find p.d.f of $u = xy$.

The JPdf

$$f(u,v) = f(x,y)\,|J| \quad \text{(formula)}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Let us assume,

$u = xy$ | $v = y$

$\frac{u}{v} = x$ | $\boxed{y = v}$

$\boxed{x = \frac{u}{v}}$

P.D. w.r.t u and v.

$\frac{\partial x}{\partial u} = \frac{1}{v}$ | $\frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = \frac{-u}{v^2}$ | $\frac{\partial y}{\partial v} = 1$ } ②

$$J = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$|J| = \frac{1}{v}$$

$$f(u,v) = f(x,y)\,|J|$$

$$= (x+y)\frac{1}{v}$$

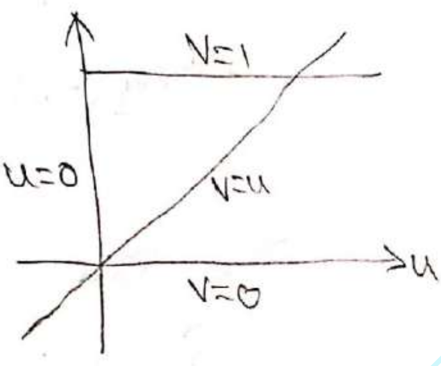
$$= \left[\frac{u}{v} + v\right]\frac{1}{v}$$

Range,

$0 \le x \le 1$ | $0 \le y \le 1$

$0 \le \frac{u}{v} \le 1$ | $0 \le v \le 1$

$0 \le u \le v$ | $0 \le v \le 1$



$0 \le u$, $u \le v$ $v < 1$

$u = v$ $v = 1$

The p.d.f 'u'

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv$$

$$= \int_{u}^{1} \left[\frac{u}{v} + v\right] \frac{1}{v} \cdot dv$$

$$= \int_{u}^{1} \left[\frac{u}{v^2} + 1\right] dv$$

$$= u \int_{u}^{1} v^{-2}\,dv + \int_{u}^{1} dv$$

$$= u\left[\frac{v^{-1}}{-1}\right]_{u}^{1} + [v]_{u}^{1} =$$

$$f(u) = 2 - 2u.$$

(*) The R.V x and y each follow an Exponential Distribution with parameter 1 ~~and~~ Are independent. find the Pdf of $u = x - y$.

If X and Y are independent random Variable Pdf is given as e^{-x}; $x \geq 0$ e^{-y}, $y \geq 0$, find the D.F of $u = \frac{x}{x+y}$, $v = x + y$ are U & V are independent.

Sln given x and y are independent,

$$f(x,y) = f(x)\,f(y)$$

$$= e^{-x} e^{-y} = e^{-(x+y)} \quad x \geq 0, \; y \geq 0$$

The JPdf

$$f(u,v) = f(x,y)\,|J|$$

where $J = \frac{\partial(x,y)}{\partial(u,v)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Given,

$u = \frac{x}{x+y}$ $\quad$ $v = x+y$

$u = \frac{x}{v}$ $\quad$ $\boxed{y = v - uv}$

$\boxed{x = uv}$

$\frac{\partial x}{\partial u} = v$ $\quad$ $\frac{\partial y}{\partial u} = -v$

$\frac{\partial x}{\partial v} = u$ $\quad$ $\frac{\partial y}{\partial v} = 1-u.$

$$J = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$= v(1-u) + vu$

$|J| = v - vu + vu$

$|J| = v.$

$\therefore f(u,v) = e^{-v}(v)$

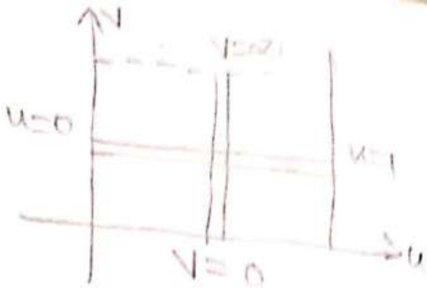
The density fn. of u,

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv.$$

Range, $x \geq 0$, $y \geq 0$

$uv \geq 0$, $v - vu \geq 0$

$u \geq 0$	$v \geq vu$
$v \geq 0$	$1 \geq u$
$v > 0,$	$1 > u, 1 = u$
$v = 0$	



The P. density fn. of u

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv$$

$$= \int_0^{\infty} ve^{-v}\,dv \quad \left[\because \int u\,dv = uv - u'v_1 + u''v_2 + \cdots\right]$$

$$= \left[v\left(\frac{e^{-v}}{-1}\right) - (1)\,e^{-v}\right]_0^{\infty}$$

$$= [0 - (0-1)] = 1. \quad \left[\because e^{-\infty} = 0,\; e^{0} = e^{-0} = 1\right]$$

The Pdf of v

$$f(v) = \int_{-\infty}^{\infty} f(u,v)\,du$$

$$= \int_0^1 ve^{-v}\,du$$

$$= ve^{-v}\,[u]_0^1$$

$$= ve^{-v}\,[1-0) = ve^{-v}!$$

Are u and v independent.

$$f(u,v) = f(u)\,f(v) = (1)\,(ve^{-v})$$

$$= ve^{-v}!$$

~~From~~ u & ~~~~ v are independent.

H/W Eg: 2.4.10

The Pdf x & y are

$f_X(x) = e^{-x}, x > 0 \quad f_Y(y) = e^{-y}, y > 0$

x & y are independent, the JPDF is

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$
$$= e^{-x} e^{-y}, \quad x, y > 0$$
$$= e^{-(x+y)}, \quad x, y > 0$$

$u = x - y \rightarrow a$

$u = x - v \rightarrow b$

$x = u + v$

$\frac{dx}{du} = 1$

$\frac{dx}{dv} = 1$

Let $v = y \rightarrow (b)$

$y = v$

$\frac{dy}{du} = 0$

$\frac{dy}{dv} = 1$

$f(x,y) = e^{-(x+y)}$

$= e^{-(u+2y)}$

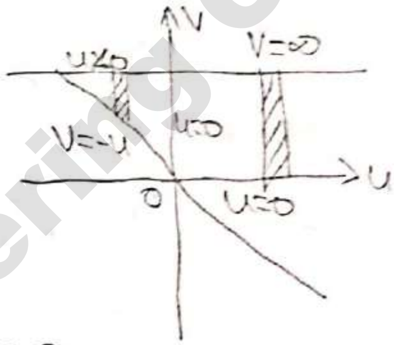
$x > 0$ | $y > 0$

$u + v > 0$ | $v > 0$

$u > -v$

The D.function of u is

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv \rightarrow (1)$$



$$f(u,v) = |J|\, f(x,y) \rightarrow (2)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} \rightarrow (3)$$

$(3) \Rightarrow J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$(2) \Rightarrow f(u,v) = e^{-(u+2v)}$

If $u < 0$ then,

$$(1) \Rightarrow f(u) = \int_{-u}^{\infty} e^{-(u+2v)}\, dv$$
$$= e^{-u} \int_{-u}^{\infty} e^{-2v}\, dv$$
$$= e^{-u} \left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty}$$
$$= e^{-u} \left[(0) - \left(\frac{e^{2u}}{-2}\right)\right]$$
$$= e^{-u} \frac{e^{2u}}{2} = \frac{e^{u}}{2}$$

If $u \geq 0$ then,

$$(1) \Rightarrow f(u) = \int_{0}^{\infty} e^{-(u+2v)}\, dv$$
$$= e^{-u} \int_{0}^{\infty} e^{-2v}\, dv$$
$$= e^{-u} \left[\frac{e^{-2v}}{-2}\right]_{0}^{\infty}$$
$$= e^{-u} \left[0 - \left(\frac{1}{-2}\right)\right]$$
$$= \frac{1}{2} e^{-u}$$

Problem based on Central limit theorem.

Model 1 :- If the average R.V follows Normal distribution $\bar{x}$ follows $N\left[\mu, \frac{\sigma}{\sqrt{n}}\right]$ by CLT, $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

The lifetime of certain kind of lights may be consider as a random variable with mean 1200 hrs. & S.D = 250 hrs find the probability avg lifetime of 60 lights exceed 1250 hrs using central limit theorem.

$\mu = E[x] = 1200$ hrs

$S.D = \sigma = 250$ hrs

$n = 60$ lights

$\bar{x}$ = Mean life time of 60 lights (or) Sample Mean.

$\bar{x}$ follows $N(\mu, \sigma/\sqrt{n})$

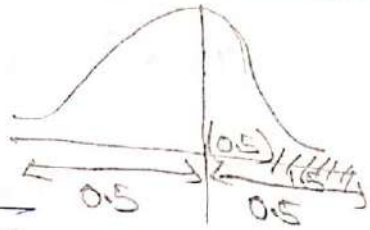
$N\left(1200, \frac{250}{\sqrt{60}}\right)$

When $[P\bar{x} > 1250]$

$z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ [formula]

When $\bar{x} = 1250$,

$$Z = \frac{1250-1200}{250/\sqrt{60}}$$



$$Z = 50 \times \frac{\sqrt{60}}{250} = \frac{\sqrt{60}}{5} = 1.55$$

$= P[Z > 1.55]$

$= 0.5 - P[0 < Z < 1.55]$

$= 0.5 - 0.439$

$= 0.061.$

A random sample of size 100 is taken from a population whose mean is 60 variance is 400. Using CLT what probability can be assert at the mean of the sample will not differ from $\mu = 60$ by more than 4.

$\mu = E[X] = 60$

Variance $= 400$

$S.D = \sigma = 20$

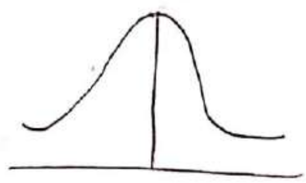
$n = 100$

$\bar{X}$ = Sample mean.

$\bar{X}$ follows $N(\mu, \sigma/n)$

$N\left(60, \frac{20}{10}\right)$

$N(60, 2)$



$P[|\bar{X}-\mu| < 4] = P[|\bar{X}-60| < 4]$

$= P[-4 < \bar{X}-60 < 4]$

$= P[56 < \bar{X} < 64]$

When $\bar{x} = 56 \Rightarrow z = \frac{56-60}{2} = \frac{-4}{2} = -2.$

When $\bar{x} = 64 \Rightarrow z = \frac{64-60}{2} = \frac{4}{2} = 2.$

$= P[-2 < z < 0] + P[0 < z < 2]$

$= P[0 < z < 2] + P[0 < z < 2]$

$= 2[P(0 < z < 2)]$

$= 2(0.477)$

$= 0.954.$

Central limit theorem:-

Statement:-

Let $x_1, x_2, \dots x_n$ be a seq. of independent identically distributed r.v with $E[x_i] = \mu$ & $Var[x_i] = \sigma^2$, $i = 1, 2, \dots n$ & if $S_n = x_1 + x_2 + \dots x_n$, then the under certain general conditions S_n follows a N.D. with mean '$n\mu$' & variance '$n\sigma^2$' as $n \to \infty$

Proof:-

a) $x_1, x_2, \dots x_n$ be 'n' independent & identically distributed r.v's

b) $E[x_1] = E[x_2] = \dots = E[x_n] = \mu$

c) $Var[x_1] = Var[x_2] = \dots = Var[x_n] = \sigma^2$

d) $S_n = x_1 + x_2 + \dots x_n.$

To prove: 1) Mean of $S_n = \mu n$. 2) $Var(S_n) = n\sigma^2$

3) S_n must be a normal variate with mean '$n\mu$' & S.d '$\sigma\sqrt{n}$'

To prove: $M_Z(t) = e^{t^2/2}$ as $n \to \infty$

where $Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$.

1) Mean of $S_n = E[S_n]$

$= E[X_1 + X_2 + X_3 + \cdots X_n]$

$= E[X_1] + E[X_2] + \cdots E[X_n]$

$= \mu + \mu + \cdots + \mu = n\mu$

2) Var of $S_n = Var(S_n) = Var[X_1 + X_2 + \cdots + X_n]$

$= Var[X_1] + Var[X_2] + \cdots + Var[X_n]$

$= \sigma^2 + \sigma^2 + \cdots + \sigma^2 = n\sigma^2$

3) Here, $Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

$M_Z(t) = E[e^{tZ}] = E\left[e^{t\left[\frac{S_n - n\mu}{\sigma\sqrt{n}}\right]}\right]$

$= E\left[e^{t\left[\frac{X_1 - \mu}{\sigma\sqrt{n}}\right]}\right]^n \rightarrow (1)$

$= E\left[1 + \frac{\left(\frac{X_1 - \mu}{\sigma\sqrt{n}}\right)t}{1!} + \frac{\left(\frac{X_1 - \mu}{\sigma\sqrt{n}}\right)^2}{2!} + \cdots\right]$

Model 2:-

If Sum of r.v follows N.D then S_n follows Random distribution

S_n follows $N(n\mu, \sigma\sqrt{n})$

$$z = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

If x_1 and $x_2 \ldots x_n$ are Poisson variant with parameter $\lambda=2$ using CLT. where $S_n = [x_1 + x_2 + x_3 + \cdots x_n]$ $n=75$ where $P[120 \leq S_n \leq 160]$ find.

Given poisson Distribution

Mean = Variance = λ

Given $\lambda = 2 = \mu$

Variance $= \sigma^2 = 2$

$\sigma = \sqrt{2}$. ↙ S.D

S_n follows $N(n\mu, \sigma\sqrt{n})$

S_n follows $N(150, \sqrt{150})$

$$z = \frac{S_n - \mu}{\sigma\sqrt{n}}$$

$$= \frac{S_n - 150}{\sqrt{150}}$$

$P(120 \leq S_n \leq 160) = ?$

When $S_n = 120 \Rightarrow z = \frac{120-150}{\sqrt{150}} = -2.45$

when $S_n = 160 \Rightarrow z = \frac{160-150}{\sqrt{150}} = 0.816.$

$= P(-2.45 \le S_n \le 0.816)$

$= P(-2.45 \le S_n \le 0) + P(0 \le S_n \le 0.816)$

$= P(0 \le S_n \le 2.45) + P(0 \le S_n \le 0.816)$

$= 0.4929 + 0.2910$

$= 0.7839.$

Unit-3

Random Variable	Random process.
A fn of possible outcomes of expariment. outcomes is mapped in $x(s)$	A fn of the possible outcomes of an expariment & also time i.e., $x(s,t)$ outcomes are mapped into wave form which is a fn of time 't'

2m Define evolutionary process:

A R.P is not stationary in any sense is called evolutionary process.

Strict sense stationary (or) strictly stationary process or Strong sense stationary :- If a R.P is stationary to all order ~~than RP~~ is said to be SSS process. [or]

UNIT-3

when $S_n = 160 \Rightarrow z = \frac{160-150}{\sqrt{150}} = 0.816.$

$= P(-2.45 \le S_n \le 0.816)$

$= P(-2.45 \le S_n \le 0) + P(0 \le S_n \le 0.816)$

$= P(0 \le S_n \le 2.45) + P(0 \le S_n \le 0.816)$

$= 0.4929 + 0.2910$

$= 0.7839.$

Unit-3

Random Variable	Random process.
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A R.P is not stationary in any sense is called evolutionary process.

Strict sense stationary (or) strictly stationary process or strong sense stationary :- If a R.P is stationary to all order ~~than RP~~ is said to be SSS process. [or]

A R.P $X(t)$ is SSS process if statistical character do not change at time $E[X(t)]$ = constant.

$Var[X(t)]$ = constant

Define wide sense stationary: (or) weakly stationary process or covariance stationary process.

A R.P is said to WSS i) if $E X(t)$ is constant. ii) $R(t_1, t_2)$ = a fn of $(t_1 - t_2)$

Problem based on SSS process:-

Standard integral formulae.

1) $\int_0^{2\pi} \cos n\theta \, d\theta = 0$

2) $\int_{-\pi}^{\pi} \cos n\theta \, d\theta = 0$

3) $\int_0^{2\pi} \cos[\omega t + n\theta] d\theta = 0$

4) $\int_{-\pi}^{\pi} \cos[\omega t + n\theta] d\theta = 0$

5) $\int_0^{2\pi} \sin n\theta \, d\theta = 0$

6) $\int_{-\pi}^{\pi} \sin n\theta \, d\theta = 0$

7) $\int_0^{2\pi} \sin[\omega t + n\theta] d\theta = 0$

8) $\int_{-\pi}^{\pi} \sin[\omega t + n\theta] d\theta = 0$

Eg: 3.2.3

If the random process $X(t)$ takes the value -1 with probability $1/3$ & takes the value 1 with probability $2/3$. Find whether $X(t)$ is a stationary process or not.

x	-1	1
P_x	$1/3$	$2/3$

$E[X(t)]$ Mean $= \sum_{i=1}^{2} x_i P(x_i)$

$= (-1)(1/3) + (1)(2/3)$

$= \frac{-1}{3} + \frac{2}{3} = 1/3.$

$Var[X(t)] = E[X^2(t)] - [E X(t)]^2$

$E[X^2(t)] = \sum_{i=1}^{2} x_i^2 P(x_i)$

$= (-1)^2 (1/3) + (1)^2 (2/3)$

$= 1/3 + 2/3 = \frac{3}{3} = 1.$

$Var[X(t)] = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}.$

Problems not SSS :-

If a R.P $X(t) = \cos(t+\phi)$

$f(\phi) = \frac{1}{\pi}, \; -\frac{\pi}{2} < \phi < \frac{\pi}{2}$ check whether the process is SSS or not

$E[X(t)] = \int_{-\pi/2}^{\pi/2} X(t) f(\phi)\, d\phi$ $\left[\because E(x) = \int_{-\infty}^{\infty} x f(x)\, dx\right]$

$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, d\phi.$

$= \frac{1}{\pi} \left[\sin(t+\phi)\right]_{-\pi/2}^{\pi/2}$

$= \frac{1}{\pi} \left[\sin\left(t+\frac{\pi}{2}\right) - \sin\, t-\pi/2\right]$

$\left[\because \sin(A+B) = \sin A \cos B + \cos A \sin B\right.$

$\left.\sin(A-B) = \sin A \cos B - \cos A \sin B\right]$

$= \frac{1}{\pi}\left[\sin t \cos\frac{\pi}{2} + \cos t \sin\frac{\pi}{2}\right] - \sin t \cos\frac{\pi}{2} - \cos t \sin\frac{\pi}{2}$

$= \frac{1}{\pi}\left[\cos t - (-\cos t)\right]$ $\left[\sin\frac{\pi}{2} = 1 \quad \cos\frac{\pi}{2} = 0\right]$

$= \frac{2\cos t}{\pi}$ fun. of t

X(t) is not SSS.

(A) 2) The process X(t) whose probability under the condition.

$$P\,X(t) = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \ldots \\ \frac{at}{1+at}, & n = 0 \end{cases}$$

show that

It is not stationary or evolutionary process

Sln

$X(t) = n$	0	1	2
P_n	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{(at)^1}{(1+at)^3}$

$$E[X(t)] = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= 0 + \frac{1(at)^0}{(1+at)^2} + \frac{2(at)^1}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}\left[1+2\left(\frac{at}{1+at}\right)+3\left(\frac{at}{1+at}\right)^2+\cdots\right]$$

$$= \frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2} = \frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{(1)^2}{(1+at)^{-2}}\right] = 1 \quad \left[\because (1)^2 = \frac{1}{(1)^2} = 1\right]$$

$$E\left[X^2(t)\right] = \sum_{n=0}^{\infty} n^2 P_n.$$

$$= \sum_{n=0}^{\infty}\left[n(n+1)-n\right]P_n = \sum_{n=0}^{\infty} n(n+1)P_n - \sum_{n=0}^{\infty} nP_n$$

$$= \sum_{n=0}^{\infty} n(n+1)\frac{(at)^{n-1}}{(1+at)^{n+1}} - 1$$

$$= \left[0+(1)(2)\frac{(at)^0}{(1+at)^2}+(2)(3)\frac{(at)^1}{(1+at)^3}+(3)(4)\frac{(at)^2}{(1+at)^4}+\cdots\right]$$

$$= \frac{2}{(1+at)^2}\left[1+3\left(\frac{at}{1+at}\right)+6\left(\frac{at}{1+at}\right)^2+\cdots\right]-1$$

$$\left[\because (1-x)^{-3} = 1+3x+6x^2+\cdots\right]$$

$$= \frac{2}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-3} = \frac{2}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-3}-1$$

$$= \frac{2}{(1+at)^2}(1+at)^3-1$$

$$= 2(1+at)-1 = 2+2at-1 = 1+2at$$

$$Var\left[X(t)\right] = E\left[X^2(t)\right]-\left[E\,X(t)\right]^2$$

$$= (1+2at)-1 = 2at$$

This process is not stationary or evolutionary.

Problem based on WSS process uniform distribution type.

1) The R.P $X(t) = A\cos(\omega t + \theta)$ is WSS if A and ω are constant & θ is uniformly distributed R.V $(0, 2\pi)$. Solution

$f(\theta) = \frac{1}{2\pi - 0}$ $0 < \theta < 2\pi$ $\left[f(x) = \frac{1}{b-a}, a < x < b\right]$

$$E[X(t)] = \int_0^{2\pi} X(t) f(\theta) d\theta$$

$$= \int_0^{2\pi} A\cos(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \int_0^{2\pi} \cos(\omega t + \theta) d\theta \quad \therefore \int_0^{2\pi} \cos(\omega t + n\theta) d\theta$$

$$= \frac{A}{2\pi} (0)$$

$E[X(t)] = 0.$

$$R(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= E[A\cos(\omega t_1 + \theta) A\cos(\omega t_2 + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega t_1 + \omega t_2 + 2\theta) + \cos(\omega t_1 - \omega t_2)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$= \frac{A^2}{2} E[\cos\omega(t_1 + t_2) + 2\theta] + \frac{A^2}{2} E\cos\omega[t_1 - t_2]$$

$\therefore E[X(t)] = \int X(t) f(\theta) d\theta$ $[\because E(c) = c]$

$$= \frac{A^2}{2} \int_0^{2\pi} \left[\cos \omega(t_1+t_2) + 2\theta\right] \frac{1}{2\pi} d\theta \Big] + \frac{A^2}{2} \cos \omega(t_1 - t_2)$$

$$= \frac{A^2}{4\pi} \int_0^{2\pi} \left[\cos \omega(t_1+t_2) + 2\theta\right] d\theta + \frac{A^2}{2} \cos \omega(t_1-t_2)$$

$\left[\because \int_0^{2\pi} \cos(\omega t + n\theta)\, d\theta = 0 .\right.$ n is integer$]$

$$R[t_1,t_2] = \frac{A^2}{2}[0] + \frac{A^2}{2} \cos \omega(t_1-t_2) = \frac{A^2}{2} \cos \omega[t_1-t_2]$$

= A function of time difference

$\{X(t)\}$ is WSS ~~property~~ process.

Problem based on WSS process [Normal distribution]

1) If $X(t) = A \cos \lambda t + B \sin \lambda t$. where A & B independent normal R.V with $E(A) = E(B) = 0$ $E(A^2) = E(B^2) = \sigma^2$ where λ is constant P.T. $\{X(t)\}$ is strict sense stationary product of order 2?

Given

$E(A) = E(B) = 0$

$E(A^2) = E(B^2) = \sigma^2 = k$ (say)

i) $E[X(t)] = E[A \cos \lambda t + B \sin \lambda t]$

$= E(A)\, E(\cos \lambda t) + E(B)\, E(\sin \lambda t)$

$= 0 + 0 = 0$

A & B are independent.

$E(AB) = E(A)\, E(B)$

ii) $R(t_1, t_2) = E[X(t_1) X(t_2)]$

$= E[A\cos\lambda t_1 + B\sin\lambda t_1][A\cos\lambda t_2 + B\sin\lambda t_2]$

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + AB\cos\lambda t_1\sin\lambda t_2 + AB\sin\lambda t_1\cos\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2]$

$= E(A^2)[\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2]$

$= K\cos[\lambda t_1 - \lambda t_2]$ $[\because \cos(A-B) = \cos A\cos B + \sin A\sin B]$

$= K\cos\lambda(t_1 - t_2)$

= fn. of time difference

$\therefore X(t)$ is WSS.

2) A R.V y is a characteristic fn. $\phi(\omega) = E(e^{i\omega y})$ = A RP defined by $X(t) = \cos(\lambda t + y)$ S.T $X(t)$ is a WSS.

Given

$\phi(\omega) = E(e^{i\omega y})$

$X(t) = \cos(\lambda t + y)$

$\phi(1) = \phi(2) = 0.$

$\phi(\omega) = E[\cos\omega y + i\sin\omega y]$

$= E[\cos\omega y] + iE[\sin\omega y]$

$\phi(1) = E[\cos y] + iE[\sin y]$

$\phi(2) = E[\cos 2y] + iE[\sin 2y]$

$0 = E[\cos y] + iE[\sin y]$

$\therefore E[\cos y] = 0$

$E[\sin y] = 0$

IIIly

$E[\cos 2y] = 0$

$E[\sin 2y] = 0$

$X(t) = \cos(\lambda t + y)$

To P.T $X(t)$ is WSS

To prove:-

1) $E\,X(t)$ is constant

2) $R(t_1,t_2)$ is fn of time difference.

1) $E[X(t)] = E[\cos(\lambda t + y)]$

$= E[\cos\lambda t \cos y - \sin\lambda t \sin y]$

$= \cos\lambda t\, E[\cos y] - \sin\lambda t\, E(\sin y)$ $[\because E(c) = 0]$

$= 0 - 0$

$E\,X(t) = 0$

2) $R(t_1,t_2) = E[X(t_1)\,X(t_2)]$

$= E[\cos(\lambda t_1 + y)\cos(\lambda t_2 + y)]$

~~$= E[\cos y] + iE[\sin y]$~~

~~$\therefore E[\cos y]$~~

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$= \frac{1}{2} E[\cos(\lambda t_1 + \lambda t_2) + 2y) + \cos(\lambda t_1 - \lambda t_2)]$

$= \frac{1}{2} E\, \cos(\lambda t_1 + \lambda t_2)\cos 2y - \sin(\lambda t_1 + \lambda t_2)\sin 2y$

$= \frac{1}{2}[\cos\lambda(t_1 - t_2)] + \frac{1}{2} E[\cos\lambda(t_1 - t_2)]$

WSS process [Discrete R.V]

Eg: 33.6 If $X(t) = Y\cos t + Z\sin t$ where Y and Z are independent R.V. Each of which assume value $-1, 2$ for probability $2/3, \frac{1}{3}$ respectively. P.T $X(t)$ is a WSS process

Y	-1	2
P(y)	2/3	1/3

Z	-1	2
P(z)	2/3	1/3

P.T $X(t)$ is WSS

To prove:- i) $E\,X(t) = \text{constant}$

ii) $R(t_1, t_2) = a$ fn time diff.

Y & Z are independent R.V.

$E[YZ] = E[Y]\,E[Z]$

$E[Y] = \sum y_i P[y_i]$

$= (-1)(2/3) + 2(1/3)$

$= 0$

$E(Y^2) = \sum y_i^2 P(y_i)$

$= (1)(2/3) + 4(1/3)$

$= 6/3 \Rightarrow 2.$

IIIly $E(Z) = 0,\ E(Z^2) = 2.$

i) $E[X(t)] = E[Y\cos t + Z\sin t]$

$= E(Y)\cos t + E(Z)\sin t$

$= 0.$

ii) $R(t_1, t_2) = E[X(t_1)\,X(t_2)]$

$= E(Y\cos t_1 + Z\sin t_1)[Y\cos t_2 + Z\sin t_2]$

$= E[Y^2\cos t_1\cos t_2 + YZ\cos t_1\sin t_2 + YZ\sin t_1\cos t_2 + Z^2\sin t_1\sin t_2]$

$= E[Y^2]\cos t_1\cos t_2 + E[YZ][\cos t_1\sin t_2 + \sin t_1\cos t_2]$

$+ E(Z^2)\sin t_1\sin t_2$

$= 2\cos t_1 \cos t_2 + 0 + 2\sin t_1 \sin t_2 \rightarrow$ by (3)(4)&(5)

$= 2[\cos t_1 \cos t_2 + \sin t_1 \sin t_2]$

$= 2[\cos(t_1 - t_2)] =$ a function of $(t_1 - t_2)$

Hence $X(t)$ is WSS process.

(5) 2m **Define Markov Process:**

A R.P $X(t)$ is said to be Markov process if $P[X(t) \le x / x(t_1) = x_1, x(t_2) = x_2 \dots x(t_n) = x_n]$

$\Rightarrow P[x(t) \le x / x(t_n) = x_n]$

Poisson process:

$$P[X(t) = x] = \frac{e^{-\lambda t} (\lambda t)^x}{\lfloor x}$$

Poisson Postulates:-

If $X(t)$ represent the no. of occurances of ~~Search~~ certain ~~time~~ event $(0, t)$ then the discrete Random process $\{X(t)\}$ is called Poisson process provided the foll. postulates are satisfied.

1) $P[1 \text{ occurrence } (t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$

2) $P[0 \text{ occurrente } (t, t+\Delta t)] = 1 - \lambda \Delta t + o(\Delta t)$

3) $P[2 \text{ or more occurrence } (t, t+\Delta t)] = 0$

4) $\{X(t)\}$ is a independent of the no. of occurrences in the event in an interval prior of

after the interval (0, t)

5) The probability of event occurs a specified no. of times in (t_0, t_0+t) depends only on t not t_0.

Ensemble average:

The ensemble avg of R.P $\{X(t)\}$ is the expected value of the R.V x at a time t. Ensemble avg $= E[x(t)]$

Ergodic process:

The R.P $x(t)$ is said to a ergodic if its ensemble avg = time average.

$$E[x(t)] = \overline{x_T}$$

where $\overline{x_T} = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt.$

whether

The first

Derive the Mean & Variance of poisson process :-

$$P[X(t)=x] = \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

Mean

$$E[X(t)] = \sum_{x=0}^{\infty} x \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x} \quad [\because E(x)=\sum x\,p(x)]$$

$$= \sum_{x=1}^{\infty} \frac{x e^{-\lambda t}(\lambda t)(\lambda t)^{x-1}}{x \lfloor x-1}$$

$$= e^{-\lambda t}(\lambda t) \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{\lfloor x-1}$$

$$= e^{-\lambda t}(\lambda t)\left[1+\frac{(\lambda t)}{\lfloor 1}+\frac{(\lambda t)^2}{\lfloor 2}+\cdots\right]$$

$$\because e^x = 1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots$$

$$= e^{-\lambda t}(\lambda t)\left[e^{\lambda t}\right]$$

Mean $= \lambda t$.

$$E[X^2(t)] = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x} \quad [\because E(x^2)=\sum x^2 p(x)]$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= e^{-\lambda t} \sum_{x=2}^{\infty} \frac{x(x-1)(\lambda t)^2(\lambda t)^{x-2}}{x(x-1)\lfloor x-2} + e^{-\lambda t} \sum_{x=1} \frac{x(\lambda t)(\lambda t)^{x-1}}{x \lfloor x-1}$$

$$= e^{-\lambda t}(\lambda t)^2 \sum_{x=2}^{\infty} \frac{(\lambda t)^{x-2}}{\lfloor x-2} + e^{-\lambda t}(\lambda t) \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{\lfloor x-1}$$

$$= e^{-\lambda t}(\lambda t)^2\left[1+\frac{\lambda t}{\lfloor 1}+\frac{(\lambda t)^2}{\lfloor 2}+\cdots\right]+e^{-\lambda t}(\lambda t)\left[1+\frac{\lambda t}{\lfloor 1}+\frac{(\lambda t)^2}{\lfloor 2}+\cdots\right]$$

$$= e^{-\lambda t}(\lambda t)^2\left[e^{\lambda t}\right]+e^{-\lambda t}(\lambda t)\left[e^{\lambda t}\right]$$

$$E[X^2] = (\lambda t)^2+\lambda t$$

$$Var[X(t)] = E[X^2(t)] - E[X(t)]^2$$

$$= \lambda^2 t^2+\lambda t-\lambda^2 t^2$$

$$= \lambda t.$$

Formula:-

$$P[X(t_1)=n_1,\ X(t_2)=n_2] = \begin{cases} \dfrac{e^{-\lambda t_2}\lambda^{n_2}\, t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{\lfloor n_1\ \lfloor n_2-n_1} & \text{if } n_2\ge n_1 \\ 0 & \text{o.w} \end{cases}$$

$$P[X(t_1)=n_1,\ X(t_2)=n_2,\ X(t_3)=n_3]$$

$$= \begin{cases} \dfrac{e^{-\lambda t_3}\lambda^{n_3}\, t_1^{n_1}(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{\lfloor n_1\ \lfloor n_2-n_1\ \lfloor n_3-n_2} & \text{if } n_3\ge n_2\ge n_1 \end{cases}$$

1) Prove that the Poisson process is Markov process.

Proof

$$P[X(t_1)=n_1,\ X(t_2)=n_2] = \begin{cases} \dfrac{e^{-\lambda t_2}\lambda^{n_2}\, t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{\lfloor n_1\ \lfloor n_2-n_1} & \text{if } n_2\ge n_1 \\ 0 & \text{o.w} \end{cases}$$

$$P(A/B) = \frac{P(A,B)}{P(B)}$$

$$P[X(t_3)=n_3 / X(t_1)=n_1, X(t_2)=n_2]$$

$$= \frac{P[X(t_3)=n_3, X(t_1)=n_1, X(t_2)=n_2]}{P[X(t_1)=n_1, X(t_2)=n_2]}$$

$$= \frac{P[X(t_1)=n_1, X(t_2)=n_2, X(t_3)=n_3]}{P[X(t_1)=n_1, X(t_2)=n_2]}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} t_1^{n_1} (t_2-t_1)^{n_2-n_1} (t_3-t_2)^{n_3-n_2}}{\lfloor n_1 \; \lfloor n_2-n_1 \; \lfloor n_3-n_2} \times \frac{\lfloor n_1 \; \lfloor n_2-n_1}{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2-t_1)^{n_2-n_1}}$$

$$= \frac{e^{-\lambda t_3} e^{\lambda t_2} \lambda^{(n_3-n_2)} (t_3-t_2)^{n_3-n_1}}{\lfloor n_3-n_2}$$

$$= \frac{e^{-\lambda(t_3-t_2)} \lambda^{(n_3-n_2)} (t_3-t_2)^{n_3-n_2}}{\lfloor n_3-n_2}$$

$$= P[X(t_3)=n_3 / X(t_2)=n_2]$$

Hence the poisson process is a Markov process.

2) The sum of 2 independent poisson process is again a poisson process.

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Proof :-

$$X(t) = X_1(t) + X_2(t)$$

$$P[X(t)=n] = \sum_{r=0}^{n} [P[X_1(t)=r]] [P[X_2(t)=n-r]]$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t} (\lambda_1 t)^r \, e^{-\lambda_2 t} (\lambda_2 t)^{n-r}}{\lfloor r \; \lfloor n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} \frac{n!}{r! \cdot (n-r)!} \cdot (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} (\lambda_1 t + \lambda_2 t)^n$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} ((\lambda_1+\lambda_2)t)^n .$$

$\Rightarrow X(t) = X_1(t) + X_2(t)$ is a P.P with parameter $\lambda_1+\lambda_2$

3) The difference of 2 independent Poisson process is not a Poisson process.

Proof:

Let $X(t) = X_1(t) - X_2(t)$

$$E[X(t)] = E[X_1(t) - X_2(t)] = E[X_1(t)] - E[X_2(t)]$$

$$= \lambda_1 t - \lambda_2 t = (\lambda_1 - \lambda_2)t$$

$$E[X^2(t)] = E[(X_1(t) - X_2(t))^2]$$

$$= E[X_1^2(t) - 2X_1(t) \cdot X_2(t) + X_2^2(t)]$$

$$= E[X_1^2(t)] - 2E[X_1(t)]\, E[X_2(t)] + E[X_2^2(t)]$$

$X_1(t)$ & $X_2(t)$ are independent.

$$= \lambda_1^2 t^2 + \lambda_1 t - 2(\lambda_1 t)(\lambda_2 t) + \lambda_2^2 t^2 + \lambda_2 t$$

$$= (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2)t$$

$$\neq (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 - \lambda_2)t$$

$X(t) = X_1(t) - X_2(t)$ is not a poisson process.

4) consider 2 consecutive occurrences of the event E_i & E_{i+1}.

$$P(T > t) = P[E_{i+1} \text{ did not occur in } (t_1, t_{i+1})]$$
$$= P(\text{No event occurs in an interval of length } t)$$
$$= P(X(t) = 0)$$
$$= e^{-\lambda t}$$

The c.d.f of T is given by

$$F(t) = P(T \le t) = 1 - P(T > t)$$
$$= 1 - e^{-\lambda t}$$

The p.d.f of T given by $f(t) = \lambda e^{-\lambda t}, t \ge 0$.

This is a exponential distribution mean $\frac{1}{\lambda}$.

Formula :-

$$P[X(t) = x] = \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{\lfloor r} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{\lfloor n-r}$$

Type 1 :-

Suppose the customer arrives at a bank according to poisson process with mean rate of 3/min. find the probability during a time interval 2 mins.

i) Exactly 4 customer arrives.
ii) More than " " ".
iii) fewer than " " in 2 mins interval

Mean rate $\lambda = 3$ per minute.

Time interval $t = 2$ per minute.

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{\lfloor n}$$

1) $\left[\begin{array}{l} P \text{ exactly 4 customer} \\ \text{2 min time interval} \end{array}\right] = P[X(2) = 4]$

$$= \frac{e^{-6}(6)^4}{\lfloor 4}$$

$$= 0.1339.$$

2) $P[\text{More than 4 customer}] = P[X(2) > 4]$

$$= 1 - P[X(2) \le 4]$$

$$= 1 - [X(2) = 0] + P[X(2) = 1] + P[X(2) = 2] + P[X(2) = 3] + P[X(2) = 4]$$

$$= 1 - \left[\frac{e^{-6}6^0}{\lfloor 0} + \frac{e^{-6}6^1}{\lfloor 1}\right] + \frac{e^{-6}6^2}{\lfloor 2} + \frac{e^{-6}6^3}{\lfloor 3} + \frac{e^{-6}}{\lfloor 4}$$

$$= 1 - e^{-6}\left[1 + 6 + \frac{36}{2} + \frac{216}{6} + \frac{1296}{24}\right]$$

$$= 0.715.$$

3) $P[\text{fewer than 4 customers}) = P[X(2) < 4]$

$$= P[X(2) = 0] + P[X(2) = 1] + P[X(2) =$$

$$= e^{-6}\left[\frac{6^0}{\lfloor 0} + \frac{6^1}{\lfloor 1} + \frac{6^2}{\lfloor 2} + \frac{6^3}{\lfloor 3}\right] = 0.1512$$

A machine force out of order whenever the components fail. The failure of this part follows P.P with a mean rate 1 per week. Find the probability that 2 weeks have elapsed since last failure

If there are 5 spare parts of this components in a, inverting and that the next supply not due to 10 weeks. Find the probability that the machine will not out of order in next 10 weeks.

S/n

$$P[X(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{\lfloor n}$$

Mean $\lambda = 1$ per week

$$P[\text{No failure 2 weeks}] = P[X(2)=0]$$

$$= \frac{e^{-(1)(2)}(2)^0}{\lfloor 0}$$

$$= e^{-2}.$$

Mean rate $\lambda = 1$

Time interval = 10 per week

n = less than or atleast 5.

ii) $P[X[10] \leq 5]$

$$= P[X[10]=0] + P(X(10)=1) + P[X(10)=2] + P[X(10)=3] + P(X(10)=4) + P[X(10)=5]$$

$$= e^{-10}\left[\frac{(10)^0}{\lfloor 0} + \frac{(10)^1}{\lfloor 1} + \frac{(10)^2}{\lfloor 2} + \frac{(10)^3}{\lfloor 3} + \frac{(10)^4}{\lfloor 4} + \frac{(10)^5}{\lfloor 5}\right]$$

$$= e^{-10}\left[1 + \frac{10}{1} + \frac{100}{2} + \frac{1000}{6} + \frac{10000}{24} + \frac{100000}{120}\right]$$

Type 1: $= 0.067$.

Eg: 3.6.9 [Refer in Book]

On the avg, a submarine on patrol sights 6 enemy ships per hour. Assuming that the no. of ships sighted in a given length of time is a poisson variate, find probability of sighting.

i) 6 ships in next half-an hour
ii) 4 ships in next 2h,
iii) atleast 1 ship in next 15 min
iv) atleast 2 ship in next 20 min.

Type 2:-

Eg: 3.6.11

A radio active source emit particle of rate 6 per minute in P.P. Emitted particle of $\frac{1}{3}$ of being recorded. find probability atleast 5 particles are record in 5-min period.

mean $\lambda = 6$

Every constant probability $p = \frac{1}{3}$.

Time interval $t = 5$

no. of recorded $n = 5$.

$$\lambda p t = (6)\left(\frac{1}{3}\right)(5) = 10$$

$$P[N(t) = n] = \frac{e^{-\lambda p t}(\lambda p t)^n}{n!}$$

P[atleast 5 particles are recorded in 5 min period] $= P[N(5) \geq 5]$

$= 1 - [P\, N(5) < 5]$

$= 1 - P[N(5)=0] + P[N(5)=1] + P[N(5)=2] + P[N(5)=3] + P[N(5)=4]$

$$= 1 - \left[\frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} \right]$$

$$= 1 - e^{-10}\left[1 + 10 + \frac{100}{2} + \frac{1000}{6} + \frac{10000}{24}\right]$$

$$= 1 - e^{-10}\left[\frac{1933}{3}\right] = 0.97075.$$

Type 3

Eg: 3.6.13

If customer arrive at a counter in accordance with a P.P with mean 3 per min. find the probability that interval b/w 2 consecutive ~~inter~~ arrival is i) more than 1 min ii) b/w 1 min & 2 min iii) 4 min or less.

Parameter $(\lambda = 3) = \int_1^\infty f(t)\, dt$ $[\because f(t) = \lambda e^{-\lambda t}]$

$f(t) = 3e^{-3t}.$

(i) $P(T > 1) = \int_1^\infty 3e^{-3x}\, dx$

$= 3\left[\frac{e^{-3x}}{-3}\right]_1^\infty \Rightarrow \left[-e^{-3x}\right]_1^\infty$

$= \left[-e^{-\infty} + e^{-3}\right] = e^{-3} \Rightarrow 0.0497$

(ii) $p(1 < T < 2) = \int_1^2 3e^{-3x}\,dx$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_1^2 \Rightarrow \left[-e^{-3x}\right]_1^2$$

$$= -\left[e^{-6} - e^{-3}\right] = e^{-3} - e^{-6}$$

$$\Rightarrow 0.04730$$

(iv) $p(T \le 4) = \int_0^4 3e^{-3x}\,dx$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_0^4 \Rightarrow -\left[e^{-3x}\right]_0^4$$

$$= -3\left[e^{-12} - e^{-0}\right] \Rightarrow \left[-e^{-12} + 1\right]$$

$$= 1 - e^{-12}$$

Markov chain:-

If $P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \ldots X_0 = a_0\}$

$\Rightarrow P\{X_n = a_n / X_{n-1} = a_{n-1}\}$ for all n then the process $\{X_n\}, n = 0,1,2,\ldots$ is called Markov chain where $a_0, a_1, \ldots a_n$ are called states of the markov chain.

One Step Transition Problem :-

The conditional probability $[X_n = a_j / X_{n-1} = a_i]$ (i.e) $P_{ij}(n-1,n)$ is called one step transition probability from state a_i to state a_j

Transition Probability Matrix :-

When, the markov chain is homogeneous the one step transition probability is denoted by P_{ij}. The matrix $P = \{P_{ij}\}$ is called TPM.

If $P_{ij}(n-1, n) = P_{ij}(m-1, m)$ then the markov chain is called homogeneous markov chain.

Chapman Kolmogorov equation:-

If P is a TPM of a homogeneous markov chain then n small n step Tpm $P^{(n)}$ is equal to P^n. (i.e) $[P_{ij}^{(n)}] = [P_{ij}]^n$

Steady State or Invariant (or) long run (or) Stationary (or) limiting State:-

π property

$\pi P = \pi$

If given 2x2 matrix

$\pi = (\pi_1, \pi_2)$

By π - property $\pi P = P$.

where $\pi_1 + \pi_2 = 1$.

If given 3x3 matrix

$\pi = (\pi_1, \pi_2, \pi_3)$

By property π, $\pi P = \pi$

where $\pi_1 + \pi_2 + \pi_3 = 1$

Problem based Steady State distribution of the chain:-

If the TPM of a markov chain is $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$ find the steady state distribution of the chain.

If $\pi = (\pi_1 + \pi_2)$ is the steady state distribution of the chain

By π property $\pi P = \pi$. where $\pi_1 + \pi_2 = 1$.

$$(\pi_1 \ \pi_2)\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [\pi_1 \ \pi_2]$$

$$\pi_1(0) + \pi_2(1/2) = \pi_1 \Rightarrow \frac{\pi_2}{2} = \pi_1 \rightarrow (1)$$

$$\pi_1(1) + \pi_2(1/2) = \pi_2 \Rightarrow \pi_1 + \frac{\pi_2}{2} = \pi_2 \rightarrow (2)$$

where, $\left[\pi_1 + \pi_2 = 1 \quad \pi_1 = 1 - \pi_2\right]$

$$\frac{\pi_2}{2} = 1 - \pi_2$$

$$\frac{\pi_2}{2} + \pi_2 = 1$$

$$\frac{3\pi_2}{2} = 1 \Rightarrow \boxed{\pi_2 = \frac{2}{3}}$$

$$(1) \Rightarrow \frac{\frac{2}{3}}{2/1} = \pi_1$$

$$\frac{2}{3} \times \frac{1}{2} = \pi_1$$

$$\boxed{\pi_1 = \frac{1}{3}}$$

Steady State markov chain $= \left(\frac{1}{3} \ \frac{2}{3}\right)$

A Sales man territory consist of 3 cities A, B & C. He never sells in same city on successive days. If he sells in A, then the next day he sells in city B. If he sells in either B or C, then the next he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of cities?

Steady state chain = (City A City B City C)

TPM

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & - \\ \frac{2}{3} & 0 & - \\ \frac{2}{3} & - & 0 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

$\frac{2}{3} + 0 + x = 1$

$x = 1 - \frac{2}{3}$

$x = \frac{3-2}{3} = \frac{1}{3}.$

$\pi = (\pi_1 \ \pi_2 \ \pi_3)$ be the steady state chain,

By π property, $\pi P = \pi$

where $\pi_1 + \pi_2 + \pi_3 = 1$.

$$\Rightarrow (\pi_1 \ \pi_2 \ \pi_3) \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & 1/2 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = (\pi_1 \ \pi_2 \ \pi_3)$$

$\Rightarrow \pi_1(0) + \frac{2}{3}\pi_2 + \frac{2}{3}\pi_3 = \pi_1 \rightarrow (1)$

$\pi_1(1) + (0)\pi_2 + \frac{1}{3}\pi_3 = \pi_2 \rightarrow (2)$

$\pi_1(0) + \frac{1}{3}\pi_2 + (0)\pi_3 = \pi_3 \rightarrow (3)$

eqn (3) $\Rightarrow \frac{\pi_2}{3} = \pi_3$

eqn (2) $\Rightarrow \pi_1 + \frac{\pi_3}{3} = \pi_2$

$\pi_1 + \frac{(\pi_2/3)}{3} = \pi_2$

$$\Rightarrow \pi_1 + \frac{\pi_2}{9} = \pi_2$$

$$\Rightarrow \pi_1 = \pi_2 - \frac{\pi_2}{9}$$

$$\boxed{\pi_1 = \frac{8}{9}\pi_2} \rightarrow (4)$$

$$\frac{8}{9}\pi_2 + \pi_2 + \frac{\pi_2}{3} = 1$$

$$\frac{8\pi_2 + 9\pi_2 + 3\pi_2}{9} = 1$$

$$\frac{20\pi_2}{9} = 1 \Rightarrow \boxed{\pi_2 = \frac{9}{20}}$$

eqn (3) $\Rightarrow \frac{\pi_2}{3} = \pi_3$

Sub. in $\pi_2 = \frac{9}{20}$ in (3)

$$\Rightarrow \frac{9/20}{3/1} = \pi_3$$

$$\frac{9}{20} \times \frac{1}{3} = \pi_3$$

$$\boxed{\frac{3}{20} = \pi_3}$$

eqn (4) $\Rightarrow \pi_1 = \frac{8}{9}\left[\frac{9}{20}\right]$

$$\boxed{\pi_1 = \frac{2}{5}}$$

Steady State chain $= \left(\frac{2}{5} \quad \frac{9}{20} \quad \frac{3}{20}\right)$

$= \left[\frac{2}{5} \times (100)\% \quad \frac{9}{20} \times (100)\% \quad \frac{3}{20} \times (100)\%\right]$

$= (40\% \quad 45\% \quad 15\%)$

problem based on Initial State P.D

If the Initial State P.D of a markov chain is $p^{(0)} = \left(\frac{5}{6} \quad \frac{1}{6}\right)$ & the TPM of the chain is $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$. find the P.D of the chain after 2 steps.

$p^{(1)} = p^{(0)} P$

$= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{12} & \frac{5}{6}+\frac{1}{12} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix}$

$p^{(2)} = p^{(1)} P$

$= \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} \frac{11}{24} & \frac{1}{12}+\frac{11}{24} \end{bmatrix}$

$= \begin{bmatrix} \frac{11}{24} & \frac{13}{24} \end{bmatrix}$

Eg:- 3.7.5

2 boys B_1, B_2 & 2 girls G_1, G_2 are throwing a ball from one to another. Each boy throws the ball to other boy with probability $\frac{1}{2}$ & to each girl $\frac{1}{4}$. On the other hand, each girl throws the ball to each boy with $\frac{1}{2}$ & never to the girl. In the long run, how often does each receive the ball?

	B_1	B_2	G_1	G_2
B_1	-	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
B_2	$\frac{1}{2}$	-	$\frac{1}{4}$	$\frac{1}{4}$
G_1	$\frac{1}{2}$	$\frac{1}{2}$	-	0
G_2	$\frac{1}{2}$	$\frac{1}{2}$	0	-

$\Rightarrow P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$

[If the tpm of a chain is stochastic matrix, then the sum of all elements of any row is equal to 1]

If $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ is steady state distribution of chain, then the property of π, we have,

$$\pi P = \pi \rightarrow (1)$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \rightarrow (2)$$

$$(1) \Rightarrow (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4) \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4)$$

$$\frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_1 \rightarrow (3)$$

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_2 \rightarrow (4)$$

$$\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 = \pi_3 \rightarrow (5)$$

$$\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 = \pi_4 \rightarrow (6)$$

In eqn (2) we change π_1, π_3 & π_4 interms of π_2 from (5) & (6) we get

$$\pi_3 = \pi_4 \rightarrow (7)$$

$$(3) - (4) \Rightarrow \frac{1}{2}\pi_2 - \frac{1}{2}\pi_1 = \pi_1 - \pi_2$$

$$\Rightarrow \frac{1}{2}\pi_2 + \pi_2 = \pi_1 + \frac{1}{2}\pi_1$$

$$\Rightarrow \frac{3}{2}\pi_2 = \frac{3}{2}\pi_1$$

$$\boxed{\pi_1 = \pi_2} \rightarrow (8)$$

$$(5) \Rightarrow \frac{1}{4}\pi_2 + \frac{1}{4}\pi_2 = \pi_3 \text{ by } (8)$$

$$\frac{1}{2}\pi_2 = \pi_3$$

i.e., $\boxed{\pi_3 = \frac{1}{2}\pi_2} \rightarrow (9)$

$(2) \Rightarrow \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$\pi_2 + \pi_2 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_2 = 1$ by (7), (8) & (9)

$\pi_2\left[1 + 1 + \frac{1}{2} + \frac{1}{2}\right] = 1.$

$\pi_2[3] = 1 \Rightarrow \pi_2 = \frac{1}{3}$

$(9) \Rightarrow \pi_3 = \frac{1}{2}\pi_2 = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

$(8) \Rightarrow \pi_1 = \pi_2 = \frac{1}{3}$

$(7) \Rightarrow \pi_4 = \pi_3 = \frac{1}{6}.$

$\therefore$ The steady state distribution of chain

$\pi = (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4)$

$= \left(\frac{1}{3} \ \frac{1}{3} \ \frac{1}{6} \ \frac{1}{6}\right)$

Thus, in the long run, the probability that each boy receives the ball $= \frac{1}{3}$, the probability that each girl receives the ball $= \frac{1}{6}$

Eg: 3.7.16

A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives 1 day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the tossed a fair die & drove to work if & only if as 6 appeared.

a) Probability that he drives to work in long run

b) Probability that he takes a train on 3rd day

State space = {Train, Car}

TPM $P = \begin{array}{c} \\ T \\ C \end{array}\begin{bmatrix} T & C \\ 0 & - \\ 1/2 & 1/2 \end{bmatrix} = \begin{array}{c} \\ T \\ C \end{array}\begin{bmatrix} T & C \\ 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$\frac{1}{6} = \{1,2,3,4,5,6\}$

P[A man travelling car] $= \frac{1}{6}$ [If 6 appear]

P[A man travell by train] $= 1 - \frac{1}{6} = \frac{6-1}{6} = \frac{5}{6}$

State space $p^{(1)} = \left[\frac{5}{6} \quad \frac{1}{6}\right]$

$p^{(2)} = p^{(1)} P$

$= \left[\frac{5}{6} \quad \frac{1}{6}\right]\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$= \left[0 + \frac{1}{2} \quad \frac{5}{6} + \frac{1}{2}\right] = \left[\frac{1}{12} \quad \frac{11}{12}\right]$

$p^{(3)} = p^{(2)} P$

$= \left[\frac{1}{12} \quad \frac{11}{12}\right]\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$= \left[0 + \frac{11}{24} \quad \frac{1}{12} + \frac{11}{24}\right] = \left[\frac{11}{24} \quad \frac{2+11}{24}\right]$

$= \left[\frac{11}{24} \quad \frac{13}{24}\right]$

P[A man travelled by train in 3rd day] $= \frac{11}{24}$

ii) Let $\pi = (\pi_1 \; \pi_2)$

By π property $\pi P = \pi$

$\Rightarrow [\pi_1 \; \pi_2]\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [\pi_1, \pi_2]$

$\pi_1(0) + \pi_2(1/2) = \pi_1 \rightarrow (1)$

$\pi_1(1) + \pi_2(1/2) = \pi_2 \rightarrow (2)$

where $\pi_1 + \pi_2 = 1$.

eqn (1) $\Rightarrow \frac{\pi_2}{2} = 1 - \pi_2$

$\frac{\pi_2}{2} + \pi_2 = 1 \Rightarrow \frac{3\pi_2}{2} = 1$

$\boxed{\pi_2 = 2/3}$

sub. in (1)

eqn (1) $\frac{\pi_2}{2} = \pi_1$

$\frac{2/3}{2} = \pi_1 \Rightarrow \frac{2}{3} \times \frac{1}{2} = \pi_1$

$\boxed{\pi_1 = \frac{1}{3}}$

$\pi = \left(\frac{1}{3} \quad \frac{2}{3}\right)$

A man travel by car long run $= \frac{2}{3}$

Eg: 3.7.21

(X) The transition probability matrix of M.C $\{x_n\}$ $n = 1, 2, 3, \ldots$ having 3 states 1, 2 & 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ & the initial distribution

is $p^{(0)} = (0.7 \quad 0.2 \quad 0.1)$ i) $P(x_2 = 3)$ & ii) $P[x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2]$

$$p^{0} = \overset{\text{St 1}}{(0.7} \quad \overset{\text{St 2}}{0.2} \quad \overset{\text{St 3}}{0.1)}$$

$P[x_0 = 1] = 0.7$, $P[x_0 = 2] = 0.2$, $P[x_0 = 3] = 0.1$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

To find $P[X_2=3]$

$$p^{(1)} = p^{(0)} P$$

$$= \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$p^{(1)} = (0.22 \quad 0.43 \quad 0.35)$$

$$p^{(2)} = p^{(1)} P$$

$$= (0.22 \quad 0.43 \quad 0.35) \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$p^{(2)} = (0.385 \quad 0.336 \quad 0.279)$$

$P[X_2=1] = 0.385$

$P[X_2=2] = 0.336$

$P[X_2=3] = 0.279.$

$$P[X_3=2, X_2=3, X_1=3, X_0=2].$$

$$= P'_{32} \, P'_{33} \, P'_{23} \, P[X_0=2]$$

$$= (0.4)(0.3)(0.2)[0.2]$$

$$= 0.0048.$$

[Eg:3.7.25] (Refer in book)

Classification of Markov chain.

Eg 3.7.13

A gambler has Rs. 2/- he bets Re. 1 at a time and wins Re. 1 with probability 1/2. He stops playing if he loses Rs. 2 or wins Rs. 4

a) What is tpm of related m-chain? b) What is the probability that he has lost his money at end of 5 plays? c) Probability that the game lasts more than 7 plays.

Soln

$\{X_n\} = \{0,1,2,3,4,5,6\}$

Since the game ends, if the player loses all the money $X_n = 2-2 = 0$ or wins Rs. 4 ($X_n = 2+4 = 6$)

a)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & - \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Refer in book.

Classification of markov chain:-

Irreducable chain:-

If $P_{ij}^{(n)} > 0$ for sum n for every i & j. then every state can be reached from every other state. This condition is satisfied the markov chain is irreducable.

Periodic State:-

If $d_i = \gcd\{m, P_{ij}^{(m)} > 0\}$

the State i is Said to be periodic if $d_i > 1$ the aperiodic if $d_i = 1$.

Note:- (1) $P_{II} > 0$, I is aperiodic

Any one state is aperiodic → All its States are periodic.

Atleast one diagonal element greater than 0.

$P_{00} = 1$ is aperiodic

$P_{01} = 0$ is aperiodic that implies all States are aperiod.

$$\begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

$P_{00} = 1$
$P_{01} = 0$
$P_{10} = 1/2$
$P_{11} = 1/2$.

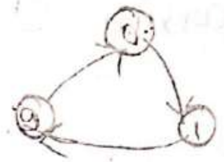
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/8 & 1/8 & 1/4 \end{bmatrix} \end{array}$$

$P_{44} = \frac{1}{4} > 0$.

State 4 is aperiodic. all States are periodic.

All the diagonal element are 0.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$



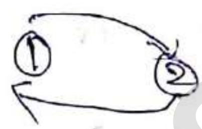
State 0 is $0 \to 1 \to 2 \to 0$ [3 steps]

State 1 in $1 \to 2 \to 0 \to 1$ "

State 2 in $2 \to 0 \to 1 \to 2$ "

All the steps are not a periodic.

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$



State 1 in $1 \to 2 \to 1$. [2 steps]

State 2 in $2 \to 1 \to 2$ [2 steps]

All its States are Periodic with period 2.

Reccurrent or Persistent State

A State i is said to be persistent or reccurrent / reccarent of the return State i that is $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$.

Non null persistant & null Persistant.

The State i said to be non null persistent if its mean reccurence time is finite $\mu_{ii} = \sum_{n=1}^{\infty} n\, f_{ii}^{(n)}$ is finite.

And null persistent $\mu_{ii} = \infty$.

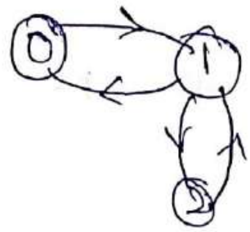
Note :-

If a markov chain is finite and irreducable than all its States are non null - persistant.

Ergodic State :-

A non Null persistant and aperiodic state is called ergodic. Find the nature of states of ~~ergodic~~ Markov chain with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



Given $X_n = \{0, 1, 2\} \rightarrow$ finite

State 0 & 1 are communicate with each other.

State 1 & 2 are " " each other

State 2 & 0 are " " " through state 1.

$\therefore$ The Markov chain is Irreducable

From 1 & 2 $\rightarrow$ all its states are non-null persistent.

State 0 in $0 \rightarrow 1 \rightarrow 0$ [2 steps]

State 1 in $1 \rightarrow 0 \rightarrow 1$ [2 steps]

State 1 in $1 \rightarrow 2 \rightarrow 1$ [2 steps]

State 2 in $2 \rightarrow 1 \rightarrow 2$ "

The states are not aperiodic with period 2.

$\therefore$ we get states of not ergodic.

Pg: 3.142 Eg: 3.7.24

UNIT-4

 A. Dhivya.

Characteristic of Queueing System

The basic chactereristic of queueing System

(1) Arrival pattern of customers
(2) Service pattern of Servers
(3) Number of Service channels
(4) System capacity
(5) queue discipline

Kendal's notation for queueing model

(a/b/c) : (d/e)

a = Probability law for the arrival (or inter arrival) time

b = probability law according ~~law~~ to which the customers are being served

c = Number of Service Stations

d = The maximum number allowed in the System (in the Service and waiting)

e = queue discipline

FIFO (or) FCFS — First in first out / First come First Served

LIFO (or) LCFS — Last - Come - First - Served

SIRO — selection in random order

PIR — priority in section

Four important models

(1) (M/M/1) : (∞/FCFS)
(2) (M/M/c) : (∞/FCFS)
(3) (M/M/1) : (N/FCFS) (or) (K/FCFS)
(4) (M/M/c) : (N/FCFS)

Notations

1.	Queue size (or) Line length	Number of customers in the system
2.	n	Number of customers in the System
3	S_n	The State in which there are 'n' customers in the system
4.	P_n	Steady State probability of having 'n' customers in the System
5.	$P_n(t)$	Transient State probability that exactly 'n' customers are in the System at time t
6.	λ	Mean arrival rate [number of arrivals per unit time]
7.	λ_n	Mean arrival rate when there are 'n' customers in the system
8.	μ	Mean Service rate [number of customers being served in unit time]
9.	μ_n	Mean Service rate where there are n customers in the system
10	L_s	Expected number of customers in the system (or) Expected queue size (waiting persons + person who is served)
11	L_q	Average number of customers in the queue [only waiting persons]
12	W_s	Average waiting time of a customer in the system
13	W_q	Expected (or) Average waiting time of a customer in the queue

Single Server Poisson queue

$P_0 = 1 - \lambda/\mu$

$P_n = \left(\lambda/\mu\right)^n \left[1 - \lambda/\mu\right]$

$P_0 = 1 - \lambda/\mu$ denotes the probability of a system being idle i.e. the system is free

$\rho = \lambda/\mu$ is called traffic intensity (or) utilization factor.

For existence of steady state solution

We should have $\rho < 1$

Average number of customers in the system (L_s) (Expected)

Let N denote the number of customers in the system $L_s = E(N)$

$$= \sum_{n=0}^{\infty} n p_n$$

$$= \sum_{n=1}^{\infty} n\left(\lambda/\mu\right)^n \left(1 - \lambda/\mu\right)$$

$$= \left(1 - \lambda/\mu\right) \sum_{n=1}^{\infty} n\left(\lambda/\mu\right)\left(\lambda/\mu\right)^{n-1}$$

$$= \left(1 - \lambda/\mu\right) \lambda/\mu \sum_{n=1}^{\infty} n\left(\lambda/\mu\right)^{n-1}$$

$$= \left(\lambda/\mu\right)\left(1 - \lambda/\mu\right)\left[1 + 2\left(\lambda/\mu\right) + 3\left(\lambda/\mu\right)^2 + \cdots\right]$$

$$= \left(\lambda/\mu\right)\left(1 - \lambda/\mu\right)\left(1 - \lambda/\mu\right)^{-2}$$

$$= \left(\lambda/\mu\right)\left(1 - \lambda/\mu\right)^{-1}$$

$$= \lambda/\mu \times \left[\frac{\mu - \lambda}{\mu}\right]^{-1} = \frac{\lambda}{\mu - \lambda}$$

$$L_s = \frac{\lambda}{\mu - \lambda} \text{ (or) } \frac{\rho}{1 - \rho} \text{ hence } \rho = \lambda/\mu$$

NOTE:

1, The average number of customers L_s in the system we mean the number of customers in the queue + the person who is getting serviced.

2, Average number of customer in the system is also denoted by $E(N_s)$

Average number of (Expected number) of customers in the queue (L_q) (or) Average length of the queue

Let N be the number of customer in the system. Then the number of customer in the queue is $(N-1)$ [excluding one being serviced]

$$L_q = E(N-1)$$

$$= \sum_{n=1}^{\infty} (n-1) p_n$$

$$= \sum_{n=1}^{\infty} (n-1)\left(1-\lambda/\mu\right)\left(\lambda/\mu\right)^n$$

$$= \left(1-\lambda/\mu\right) \sum_{n=1}^{\infty} (n-1)\left(\lambda/\mu\right)^n$$

$$= \left(1-\lambda/\mu\right) \sum_{n=1}^{\infty} (n-1)\left(\lambda/\mu\right)^2\left(\lambda/\mu\right)^{n-2}$$

$$= \left(1-\lambda/\mu\right)\left(\lambda/\mu\right)^2 \sum_{n=2}^{\infty} (n-1)\left(\lambda/\mu\right)^{n-2}$$

[When $n=1$, the 1st term becomes zero]

$$= \left(1-\lambda/\mu\right)\left[\lambda/\mu\right]^2\left[1+2\left(\lambda/\mu\right)+3\left(\lambda/\mu\right)^2+\cdots\right]$$

$$= \left(1-\lambda/\mu\right)\left[\lambda/\mu\right]^2\left[1-\lambda/\mu\right]^{-2}$$

$$= \frac{1}{1-\lambda/\mu}\left[\lambda/\mu\right]^2 = \frac{\mu}{\mu-\lambda} \times \frac{\lambda^2}{\mu^2}$$

$$\boxed{L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}} \quad (\text{or}) \quad \frac{\rho^2}{1-\rho}, \quad \rho = \lambda/\mu$$

5

Average (Expected) number of customer in nonempty queue (L_w)

[Average length of the queue formed from time to time]

Since the non empty queue is non empty, the number of customer in the system must be atleast 2 one in the queue and one being served.

$$L_w = E\{(N-1)/(N-1)>0\}$$

$$= \frac{E(N-1)}{P((N-1)>0)} = \frac{E(N-1)}{P(N\geq 1)}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \times \frac{1}{\sum_{n=2}^{\infty} P_n}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \times \frac{1}{\left(\frac{\lambda^2}{\mu^2}\right)}$$

$$\sum_{n=2}^{\infty} P_n = (\lambda/\mu)^2 P_0 + (\lambda/\mu)^3 P_0 + \cdots$$

$$= (\lambda/\mu)^2 P_0 \left[1 + (\lambda/\mu) + (\lambda/\mu)^2 + \cdots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^2 (1-\lambda/\mu)(1-\lambda/\mu)^{-1}$$

$$= (\lambda/\mu)^2$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \times \frac{\mu^2}{\lambda^2} = \frac{\mu}{\mu-\lambda}$$

$$\boxed{L_w = \frac{\mu}{\mu-\lambda}}$$

probability that the number of customers in the system exceeds k

$$P(\text{number of customers in the system} > k) = (\lambda/\mu)^{k+1}$$

NOTE: p(number of customers in the queue) > k) = $(\lambda/\mu)^k$

[Here we are excluding the person who is in Service]

* probability that the system is busy = $1 - p_0 = \lambda/\mu$
* probability that the system is empty = $p_0 = 1 - \lambda/\mu$
* Expected (Average) waiting time of a customer in the system (W_s)

$$W_s = \frac{1}{\mu - \lambda}$$

* probability that the waiting time of a customers in the system Exceeds t

$$P(W_s > t) = e^{-(\mu-\lambda)t}$$

* probability that the waiting time in the queue Exceed t $P(W_q > t) = \lambda/\mu \, e^{-(\mu-\lambda)t}$
* Average (Expected) waiting time of a customer in the queue (W_q) = $\frac{\lambda}{\mu(\mu-\lambda)}$
* Average waiting time of a customer in the queue if he has to wait (W_w)

$$W_w = \frac{1}{\mu - \lambda}$$

Equation of Birth and death process

[A/M - 2017] [A/M - 2015]

Let $N(t)$ denote the total number of individuals at epoch t starting from $t=0$

consider the interval 0 to $t+h$

suppose it is split upto 2 periods 0 to t and $t+h$

The event $\{N(t+h) = n, n \geq 1\}$ having the probability $p_n(t+h)$ can occur in a number of mutually exclusive ways

Birth μ or more than one death between t and $t+h$ such an event is $o(h)$

There will remain four other events to be considered

A_{ij} : $(n-i+j)$ individuals by epoch t, i-birth and j-death between t and $t+h$ $i,j = 0,1$

Let $p_n(t) = P[N(t)=n]$ Then

$$P(A_{00}) = p_n(t)[1-\lambda_n h + o(h))(1-\mu_n h + o(h))$$

$$= p_n(t)[1-(\lambda_n+\mu_n)h + o(h)]$$

$$P(A_{10}) = p_{n-1}(t)(\lambda_{n-1}h + o(h))(-\mu_{n-1}h + o(h))$$

$$= p_{n-1}(t)[\lambda_{n-1}h + o(h)]$$

$$P(A_{01}) = p_{n+1}(t)(1-\lambda_{n-1}h + o(h))(\mu_{n+1}h + o(h))$$

$$= p_{n+1}(t)[\mu_{n+1}h + o(h)]$$

and $P(A_{11}) = p_n(t)(\lambda_n h + o(h))(\mu_n h + o(h))$

$$= o(h)$$

$\therefore$ for $n \geq 1$ we have

$$p_n(t+h) = p_n(t)[1-(\lambda_n+\mu_n)h] + p_{n-1}(t)[\lambda_{n-1}h] + p_{n+1}(t)\mu_{n-1}(h) + o(h)$$

$$\frac{p_n(t+h)-p_n(t)}{h} = -(\lambda_n+\mu_n)p_n(t) + p_{n-1}(t)\lambda_{n-1} + p_{n-1}(t)\mu_{n+1} + \frac{o(h)}{h}$$

As $h \to 0$ we have

$$p_n'(t) = -(\lambda_n+\mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + p_{n-1}(t)\mu_{n+1} \leq 1 \quad \text{--- (1)}$$

For $n=0$, we have

$$p_n'(t) = -(\lambda_n+\mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + p_{n-1}(t)\mu_{n+1} \leq 1$$

$$p_0(t+h) = p_0(t)[1-(\lambda_0+\mu_0)h] + p_1(t)\mu_1 h + o(h)$$

$$\frac{p_0(t+h)-p_0(t)}{h} = -p_0(t)[\lambda_0+\mu_0] + p_1(t)\mu_1 + \frac{o(h)}{h}$$

As $h \to 0$, we have

$$p_0'(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t) \quad \text{--- (2)}$$

If at epoch $t=0$ there were $i\ (\geq 0)$ individuals, then the initial condition is

$$p_n(0) = 0, \quad n \neq 1; \quad p_1(0) = 1$$

The equations (1) and (2) are known as equations of birth and death process

Terminology

$\lambda \to$ Mean arrival rate $= \frac{1}{\text{Mean arrival time}}$

$\mu \to$ Mean service rate $= \frac{1}{\text{Mean service time}}$

$c^{(or)}_{s} \to$ Number of servers

$\rho \to$ traffic intensity / utilisation factor / busy period

$P_0 \to P$[No customer in the system / idle]

$P_n \to P$[n customer in the system]

$$P[(N=n) = P_n]$$

$L_s \to$ Average or Expected number of customer in the system

$L_q \to$ Average number of customers in the queue / Average length of the queue

$W_s \to$ Average waiting time of a customer in the system

$W_q \to$ Average waiting time of a customer in the queue

Pure Birth process The arrival process assumes that the customers arrive at the queueing system and never leave the system. Such a process is called pure birth process

Pure ~~Birt~~ Death process

The departure process assumed that no customers joins the system while the service is continued for those who are already in the system

Birth and death process

The simultaneous occurence of arrivals and departures is also called Birth-and-Death Process.

Model I [M/M/1] : [∞/FCFS] (or) [∞/FIFO]

$\lambda_n = \lambda$ and $\mu_n = \mu$ $[\lambda < \mu]$

$\rho = \frac{\lambda}{\mu}$

$P_0 = 1 - \rho$

$P_n = \rho^n P_0$, $n = 0, 1, 2 \ldots$

Little's formula

$L_s = \frac{\rho}{1-\rho}$

$L_q = L_s - \lambda/\mu$

$W_s = \frac{1}{\lambda} L_s$

$W_q = \frac{1}{\lambda} L_q$

1. Average no. of customers in non-empty queue L_w $= \frac{1}{1-\rho}$
2. The probability that the no. of customers in system exceed k $= P(N > k) = \rho^{k+1}$
3. The probability that the queue is non empty $= P(N > 1) = \rho^2$
4. The probability that the waiting of customer in the system exceed t $=$ $P[w_s > t] = e^{-(\mu-\lambda)t}$
5. The probability density function of waiting time in the system $f(w) = (\mu-\lambda)e^{-(\mu-\lambda)w}$
6. Density function of waiting time in the queue $g(w) = \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)w}$
7. Probability of arriving customer has to wait $E[w_q / w_q > 0] = \frac{1}{\mu-\lambda}$
8. Probability arriving customer has to wait $P[n=1]$ (or) $P[n>0] = \rho$.

(1) A supermarket has a single cashier. during peak hours. customers arrive at a rate of 20 per hour. The average number of customers that can be processed by the cashier is 24 per hour find (a) The probability that the cashier is idle

(ii) The average number of customers in the queueing system

(iii) The average time of customers spend in the system

(iv) The average number of customers in the queue

(v) The average time a customers spends in the queue waiting for service

Solution Single cashier → single server [1]

customers → infinite capacity [∞]

The given problem is (M/M/1) : (∞/FCFS) (or) (∞/FIFS)

Mean arrival rate $(\lambda) = 20$ per hour

Mean service rate $(\mu) = 24$ per hour

$$\rho = \frac{\lambda}{\mu} = \frac{20}{24}$$

(1) The probability that the cashier is idle

$$P_0 = 1-\rho$$

$$= 1 - \frac{20}{24} = \frac{24-20}{24} = \frac{4}{24}$$

(2) The average number of customer in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{\frac{20}{24}}{1-\frac{20}{24}} = \frac{\frac{20}{24}}{\frac{4}{24}}$$

$$= \frac{20}{4} = 5$$

(3) The average time a customer spend in the system $W_s = \frac{1}{\lambda} L_s$

$$= \frac{1}{20}(5)$$

$$= \frac{1}{4} h$$

The average number of customers in the queue

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= 5 - \frac{20}{24}$$

$$= \frac{25}{6}$$

$$= 4.1667$$

The average time a customer spends in the queue waiting for service

$$W_q = \frac{1}{\lambda} L_q$$

$$= \frac{1}{20}(4.1667)$$

$$= 0.2083 h$$

---

(2) Customers arriving at a watch repair shop according to poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes

(1) Find the average number of customers L_s in the shop

(2) Find the average number of customers spends in the shop W_s

(3) Find the average number of customers in the queue (

(4) What is the probability that the server is idle

[M/J - 2016]

Solution

Watch repair shop → Single server [1]

Customers → Infinite capacity [∞]

(13)

Given Mean inter arrival time = 10 minute

∴ Mean arrival rate $(\lambda) = \frac{1}{\text{Mean arrival time}}$

$= \frac{1}{10}$ per minute

Mean service rate $(\mu) = \frac{1}{\text{Mean Service time}}$

$= \frac{1}{8}$ per minute

$$\rho = \frac{\lambda}{\mu} = \frac{1/10}{1/8} = \left(\frac{8}{1}\right)\left(\frac{1}{10}\right) = \frac{4}{5}$$

$$P_0 = 1-\rho$$
$$= 1 - 4/5 = \frac{1}{5}$$

Average number of customers in the shop

$$L_s = \frac{\rho}{1-\rho} = \frac{4/5}{1-4/5} = \frac{4/5}{1/5} = 4$$

Average ~~number of~~ time a customer spends in the shop

$$W_s = \frac{1}{\lambda} L_s = \frac{1}{\left(\frac{1}{10}\right)} 4$$

$$= 4(10) = 40.$$

Average number of customer in the queue

$$L_q = L_s - \lambda/\mu$$
$$= 4 - 4/5 = \frac{20-4}{5} = \frac{16}{5} = 3.2$$

$P[\text{Server is idle}] = P_0 = \frac{1}{5}$

Extra $W_q = \frac{1}{\lambda} L_q$

$$= \frac{1}{\left(\frac{1}{10}\right)}(3.2) = (10)(3.2) = 32$$

(14)

③ A T.V repairman finds that the time spent on his Job has an exponential distribution with mean 30 minutes. If he repair sets in the order in which they come, which follows a Poisson arrival pattern with average rate of 10 per 8 hrs day. ① what is the repairman's expected idle time each day

② How many Jobs are ahead of an average set brought in ?

③ What is the average number of Jobs in a non empty queue ?

Solution A. T.V repairman → single server [1]

sets → infinite capacity [∞]

∴ The given problem is $(M/M/1):(\infty/FCFS)$

Given Mean arrival rate $(\lambda) = 10$ per (8 hr) day

Mean service time $= 30$ minutes $= \frac{1}{2}$ hr

$$\text{Mean Service rate } (\mu) = \frac{8}{\text{Mean service time}}$$

$$= \frac{8}{1/2} = 16 \text{ set per (8 hr) day}$$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{16} = \frac{5}{8}$$

$$P_0 = 1-\rho = 1-\frac{5}{8} = \frac{8-5}{8} = \frac{3}{8}$$

$$L_s = \frac{\rho}{1-\rho} = \frac{5/8}{1-5/8} = \frac{5/8}{3/8} = \frac{5}{3}$$

① $P[\text{repairman is idle}] = P_0 = 3/8$

The expected idle time $= 8 \times 3/8 = 3$ hrs

② Average number of Jobs ahead of an average set brought in $L_s = \frac{5}{3}$

$$L_s = \frac{\rho}{1-\rho} = \frac{5/8}{1-5/8} = \frac{5/8}{\frac{8-5}{8}}$$

$$= \frac{5}{3}$$

(3) Average number of Jobs in a non empty queue (15)

$$L_w = \frac{1}{1-\rho}$$

$$= \frac{1}{1-5/8} = \frac{1}{\frac{8-5}{8}} = \frac{8}{3} = 2.667$$

---

(4) Customers arrive at one-man barbershop according to a poisson process with a mean interarrival time of 12 min. customers spend an average of 10 min. in the barber's chair

(1) What is the expected number of customers in the barber shop and in the queue

(2) calculate the % of time of arrival, can walk straight into the barber's chair without having to wait

(3) How many time can customer except to spend in the barber's shop

(4) Management will provide another chair and hire another barber, when a customer's waiting time in the shop exceeds 1.25h. How much must the average rate of arrivals increase to warrant a second barber

(5) What is the average time customers spend in the queue

(6) What is the probability that the waiting time in the system is greater than 30 min

(7) calculate the % of customers who have to wait prior to getting into the barber's chair.

(8) what is the probability that more than 3 customers are in the system?

Solution

Given one man barber shop → Single server [1]

customers → Infinite capacity [∞]

∴ The given problem is $[M/M/1]:[\infty/FCFS]$ (or $[\infty/FIFO]$)

Mean arrival time = 12 minutes

$$\text{Mean arrival rate } (\lambda) = \frac{1}{\text{Mean arrival time}}$$

$$= \frac{1}{12} \text{ per minute}$$

Mean service time = 10 minutes

$$\text{Mean service rate } (\mu) = \frac{1}{\text{Mean service time}}$$

$$= \frac{1}{10} \text{ per minutes}$$

$$\Rightarrow \rho = \frac{\lambda}{\mu} = \frac{1/12}{1/10} = \frac{10}{12} = 5/6$$

(1) The expected number of customers in the system [barber shop]

$$L_s = \frac{\rho}{1-\rho} = \frac{5/6}{1-5/6} = \frac{5/6}{1/6} = 5$$

The expected number customer in the queue

$$L_q = L_s - \lambda/\mu$$

$$= 5 - 5/6 = \frac{30-5}{6} = \frac{25}{6} = 4.17$$

$$L_q = 4.17$$

(2) P[A customer walk straight into the barber's chair without having to wait]

= P[No customer in the system]

$$\Rightarrow P_0 = 1-\rho = 1 - 5/6 = \frac{6-5}{6} = 1/6$$

$$P_0 = 0.1667$$

∴ The percentage of time of arrival

$= (0.1667) \times 100 = 16.67\%$

(3) Expected time a customer spends in the (barbershop) system

$W_s = \frac{1}{\lambda} L_s$

$= \frac{1}{(1/12)} (5) = (12)(5) = 60$

(4) Given $W_s > 1.25h$

$[1.25h = (1.25)(60) \text{ min}$

$= 75 \text{ min}]$

$\Rightarrow W_s > 75 \text{ min}$

$\left[W_s = \frac{1}{\lambda} L_s = \frac{1}{\lambda} \frac{\rho}{1-\rho} = \frac{1}{\lambda} \left[\frac{\lambda/\mu}{1-\lambda/\mu} \right] = \frac{1}{\mu-\lambda} \right]$

Here $W_s = \frac{1}{\mu - \lambda_R}$, where $\lambda_R \rightarrow$ required arrival rate

$\therefore \frac{1}{\mu - \lambda_R} > 75 \Rightarrow \frac{1}{75} > \mu - \lambda_R$

$\Rightarrow \lambda_R > \mu - \frac{1}{75}$

$\Rightarrow \lambda_R > \frac{1}{10} - \frac{1}{75}$ $\quad \left[\because \mu = \frac{1}{10}\right]$

$\Rightarrow \lambda_R > \frac{13}{150}$

Hence, the arrival rate should increase by

$\frac{13}{150} - \frac{1}{12} = \frac{1}{300}$ per minute

(5) Average waiting time per customer in the queue

$W_q = \frac{1}{\lambda} L_q = \frac{1}{(1/12)} (4.17)$

$= (12)(4.17) = 50$

(6) P[waiting time in the system > 30 minutes]

$= P[W > 30]$

[Formula $P[W > t] = e^{-(\mu-\lambda)t}$]

$= e^{-\left(\frac{1}{10} - \frac{1}{12}\right)30}$

$= e^{-\left(\frac{12-10}{120}\right)30}$

$= e^{-0.5} = 0.6065$

(7) P[a customer has to wait] $= 1 - P_0 = \rho = \frac{5}{6}$

∴ The percentage of customers who have to wait $= \frac{5}{6} \times 100$

$= \frac{500}{6} = 83.33$

(8) P[More than 3 customers in the system]

Formula $P[N > k] = \rho^{k+1}$

$= P[N > 3]$

$= \rho^{3+1} = \rho^4$

$= \left(\frac{5}{6}\right)^4 = 0.4823$

(5) Arrivals at a telephone booth are considered to be poisson will an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min

(1) Find the average number of persons waiting in the system

(2) What is the probability that a person arriving at the booth will have to wait in the queue

(3) What is the probability that it will take him more than 10 min altogether to wait for the phone and complete his call

(4) (a) Estimate the fraction of the day when the phone will be in use (b) Average queue length is formed time to time

(5) The telephone department will install a second booth, when convinced that an arrival has to wait on the average for atleast 3 min for phone. By how much flow of arrivals should increase in order to justify a second booth

Solution Given Telephone booth → Single server [1]

Arrival at a telephone booth → Infinite capacity [∞]

The given problem is $(M/M/1) : (\infty / FCFS)$ (or) (FIFO)

Mean arrival time = 12 minutes

$$\text{Mean arrival rate} = \frac{1}{\text{Mean arrival time}} = \frac{1}{12} \text{ per min}$$

Mean service time = 4 minutes

Mean service rate (μ) = $\frac{1}{\text{mean service time}}$

$= \frac{1}{4}$ per minute

$\rho = \frac{\lambda}{\mu} = \frac{1/12}{1/4} = \left(\frac{1}{12}\right)(4) = 1/3$

(1) Average number of persons waiting in the system

$L_s = \frac{\rho}{1-\rho} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = 1/2$

(2) probability that the person arriving in the booth has to wait in the queue

$P[N \geq k] = \rho^k$ [formula]

$P[N \geq 1] = \rho^1 = 1/3 = 0.3333$

(or)

$P[W > 0] = 1 - P[W = 0]$

$= 1 - P[\text{no customer in the system}]$

$= 1 - P_0 = 1 - [1-\rho] = \rho = 1/3$

(3) A person takes more than 10 minutes to wait and complete this call

$P[W > t] = e^{-(\mu-\lambda)t}$ [formula]

$P[W > 10] = e^{-\left(\frac{1}{4} - \frac{1}{12}\right)10}$

$= e^{-\left[\frac{12-4}{48}\right]10}$

$= e^{-\left[\frac{8}{48}\right]10}$

$= e^{-5/3} = 0.1889$

(4) (a) $P[\text{Phone in use}] = P[\text{Phone is busy}]$

$= 1 - P[\text{phone is idle}]$

$= 1 - P_0 = 1 - [1-\rho]$

$= \rho = 1/3$

(5) The second phone will be installed if $W_q > 3$

$$\left[W_q = \frac{1}{\lambda} L_q = \frac{1}{\lambda}\left[L_s - \lambda/\mu\right] \right.$$

$$= \frac{1}{\lambda}\left[\frac{\rho}{1-\rho} - \frac{\lambda}{\mu}\right]$$

$$= \frac{1}{\lambda}\left[\frac{\frac{\lambda}{\mu}}{1-\lambda/\mu} - \frac{\lambda}{\mu}\right]$$

$$= \frac{1}{\lambda}\left[\frac{\frac{\lambda}{\mu}}{\frac{\mu-\lambda}{\mu}} - \lambda/\mu\right]$$

$$= \frac{1}{\lambda}\left[\frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu}\right] = \frac{1}{\mu-\lambda} - \frac{1}{\mu}\Bigg]$$

Here $W_q = \dfrac{1}{\mu - \lambda_R} - \dfrac{1}{\mu}$ [where $\lambda_R \to$ required arrival rate]

$$\therefore \frac{1}{\mu-\lambda_R} - \frac{1}{\mu} > 3$$

$$\Rightarrow \frac{1}{\mu-\lambda_R} > 3 + \frac{1}{\mu}$$

$$\Rightarrow \frac{1}{\frac{1}{4}-\lambda_R} > 3 + \frac{1}{(1/4)}$$

$$\Rightarrow \frac{1}{\frac{1}{4}-\lambda_R} > 7$$

$$\Rightarrow \frac{1}{7} > \frac{1}{4} - \lambda_R$$

$$\Rightarrow \lambda_R > 1/4 - 1/7$$

$$\Rightarrow \lambda_R > \frac{3}{28}$$

(4) (b) The average queue length formed from time to time is given by

$$L_w = \frac{1}{1-\rho} = \frac{1}{1-1/3}$$

$$= \frac{1}{\frac{3-1}{3}}$$

$$L_w = 3/2$$

Hence the arrival rate should increase by $\dfrac{3}{28} - \dfrac{1}{12} = \dfrac{1}{42}$ per minutes

(2.21)

(6) In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes, calculate

(1) The mean queue size

(2) The Probability that the queue size exceeds 10

(3) If the input of trains increases to an average of 33 per day what will be change in the above quantities

Solution: A railway marshalling yard → single server [1]

goods trains → infinite capacity [∞]

∴ The given problem is $(M/M/1):(\infty/FIFO)$

(or) $(M/M/1):(\infty/FCFS)$

Mean arrival rate (λ) = 30 trains per day

Mean Service time = 36 minutes

$$\text{Mean Service rate}(\mu) = \frac{1}{\text{Mean Service time}}$$

$$= \frac{(24)(60)}{36} = \frac{1440}{36}$$

= 40 trains per day

$$\rho = \frac{\lambda}{\mu} = \frac{30}{40} = \frac{3}{4}$$

$$P_0 = 1-\rho = 1-\frac{3}{4} = \frac{4-3}{4} = \frac{1}{4}$$

$$L_s = \frac{\rho}{1-\rho} = \frac{3/4}{1-3/4} = \frac{3/4}{1/4} = 3$$

$$L_q = L_s - \frac{\lambda}{\mu} = 3 - \frac{3}{4} = \frac{9}{4}$$

$$W_s = \frac{1}{\lambda} L_s = \frac{1}{30}(3) = \frac{1}{10}$$

$$W_q = \frac{1}{\lambda} L_q = \frac{1}{30}\left(\frac{9}{4}\right) = \frac{3}{40}$$

(i) Mean queue size
= Mean queue length
$= L_q = \frac{9}{4} = 2.25$ trains

(2) P(queue size Exceeds 10)
$= P(N>10) = \rho^{10+1} = \rho^{11} = \left(\frac{3}{4}\right)^{11}$
$= 0.0422$ $\left[\because P[N>n] = \rho^{n+1}\right]$

P[queue size is atleast 10]
$= P[N \geq 10] = \rho^{10} = \left(3/4\right)^{10} = 0.0563$
$\left[\because P[N \geq n] = \rho^{n}\right]$

(iii) If the input of trains increases to an average of 33 per day

Here $\lambda = 33$, $\mu = 40$

$\rho = \lambda/\mu = \frac{33}{40}$

$L_s = \frac{\rho}{1-\rho} = \frac{33/40}{1-33/40} = \frac{33}{7}$

$L_q = L_s - \lambda/\mu = \frac{33}{7} - \frac{33}{40} = \frac{1089}{280}$
$= 3.889 = 4$ trains

$\therefore$ change in average queue size $= 4 - 2.25$
$= 1.75$ trains

Also, P[queue size exceeds 10] $= P[N>10]$

$= \rho^{10+1} = \rho^{11} = \left(\frac{33}{40}\right)^{11}$ [$\because$ exceed $N>10 = \rho^{n+1}$, atleast $N \geq 10 = \rho^{n}$]

$= 0.1205$

Thus increase required Probability
$= 0.1205 - 0.0422$
$= 0.0783$

Model – III [M/M/1] : [N / FIFO (or) FCFS]

$\rho = \lambda/\mu$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}, \quad \lambda \neq \mu$$

$$= \frac{1}{N+1}, \quad \lambda = \mu$$

$$P_n = \rho^n P_0, \quad \lambda \neq \mu$$

$$= \frac{1}{N+1}, \quad \lambda = \mu$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$L_q = L_s - \lambda'/\mu$ (or) $L_q = L_s - (1-P_0)$
$= L_s - 1 + P_0$

$W_s = \frac{1}{\lambda'} L_s$
$W_q = \frac{1}{\lambda'} L_q$ [Average waiting time in the system and in the queue]

$\lambda' = \mu(1-P_0) = \lambda(1-P_n)$

Fraction of potential customer lost
= fraction of time system is full

$P_N = P_0 \rho^N$

(25)

(4) A one-person barbershop has six chairs to accommodate people waiting for hair cut. Assuming customers who arrive when all six chairs are full, leave without entering the barbershop. customers arrive at the average rate of 3 per hour and spend an average of 15 minutes in the barbershop. Then find

(a) The probability that a customer can get directly into the barber chair upon arrivals

(b) Expected number of customers waiting for haircut.

(c) Effective arrival rate

(d) The time a customer can expect to spend in the barbershop

Solution

one barber → single channel

chairs → 6 (finite)

Model → (M/M/1) : (N / FCFS)

Mean arrival rate $\lambda = 3$ per hour

Mean service time $\frac{1}{\mu} = 15$ min

Mean service rate $\mu = \frac{1}{15}$ per minute

$= 4$ per hour

N = capacity of the system

= 6 waiting and 1 getting haircut

$= 6 + 1$

$= 7$

(26)

(a) The customers will directly go into the barber chair when the system at the time of his arrival is empty

P(There is no one in the barbershop) = P_0

$$= \frac{1-\rho}{1-\rho^{N+1}} \quad \left[\rho = \lambda/\mu = 3/4\right]$$

$$= \frac{1-3/4}{1-(3/4)^{7+1}}$$

$$= \frac{1-3/4}{1-(3/4)^8}$$

$$= 0.2778$$

(b) Expected number of customers waiting for the hair cut (L_q)

$$L_q = L_s - 1 + P_0 = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-(\rho)^{N+1}} - 1 + P_0$$

$$= \frac{3/4}{1-3/4} - \frac{(7+1)(3/4)^{7+1}}{1-(3/4)^{7+1}} - 1 + 0.2778$$

$$= 1.38$$

(c) Effective arrival rate $\lambda' = \mu(1-P_0)$

$$= 4(1-0.2778)$$

$$= 2.89 \text{ per hour}$$

(d) Average waiting time of a customer in the system $W_s = \frac{L_s}{\text{Effective arrival rate}}$

$$= \frac{L_q + 1 - P_0}{\lambda'}$$

$$= \frac{1.38 + 1 - 0.2778}{2.89} = 43.6 \text{ minu}$$

(3) Patients arrive at a clinic having single doctor according to poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients Examination time per patient is exponential with mean rate of 20 per hour

(a) Find the effective arrival rate at the clinic

(b) what is the probability that an arriving patient will not wait

(c) What is the expected waiting time until a patient is discharged from the clinic [A/M - 2018], [N/D-2015]

Solution only one doctor → single service

14 can be accommodate

↳ finite capacity

Model → (M/M/1):(N/FCFS)

Arrival rate $\lambda = 30$ per hour

Service rate $\mu = 20$ per hour

N = capacity of the system

= chairs to accommodate waiting people + one chair in service

$= 14+1 = 15$

$$\rho = \frac{\lambda}{\mu} = \frac{30}{20} = \frac{3}{2} = 1.5$$

(i) Effective arrival rate $= \mu(1-P_0)$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-1.5}{1-(1.5)^{16}} = \frac{-0.5}{(1-1.5)^{16}}$$

$= 0.00076$

Effective arrival rate $= \mu(1-p_0)$
$= 20[1-0.00076] = 19.98$
$= 20$ per hour (approx)

(b) P[a patient will not wait] $= p_0$
$= 0.00076$

(c) $L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$

$= -3 - \frac{16 \times (1.5)^{16}}{1-(1.5)^{16}} = 13$ Patients [approx]

$= -3 +$

Expected waiting time $W_s = \frac{L_s}{\text{Effective arrival rate}}$

$= \frac{13}{20} = 0.65$ hour

$= 39$ minutes

(4) In a single server queueing system with Poisson input and exponential services times, If the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hour and the maximum Possible number of calling units in the system is 2.

Find (1) P_n $(n \geq 0)$

(2) Average number of calling units in the system.

Solution This model $(M/M/1):(N/FCFS)$

Arrival rate $\lambda = 3$ units/hour

$\frac{1}{\mu} = 0.25$

∴ service rate $\mu = 4$ units/hour

Now $\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

$\Rightarrow 1-\rho = 1-0.75 = 0.25$

Capacity of the system $N=2$

① $P_n (n \geq 0)$

$$P_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}}$$

$$= \frac{(0.25)(0.75)^n}{1-(0.75)^3}$$

$$= (0.43)(0.75)^n$$

We know that $L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$

$$= \frac{0.75}{1-0.75} - \frac{3(0.75)^3}{1-(0.75)^3}$$

$$L_s = 0.81$$

(or)

$$L_s = \sum_{n=0}^{N} n p_n = \sum_{n=0}^{2} n(0.43)(0.75)^n$$

$$= (0.43)\sum_{n=0}^{2} n(0.75)^n$$

$$= 0.43\left[0 + 0.75 + 2(0.75)^2\right]$$

$$L_s = 0.81$$

(30)

(5) A car park contains 5 cars. The arrival of cars follows a poisson distribution at a mean rate of 10 per hour. The length of time each car spends in the car park is exponentially distributed with mean of 2 minutes.

(1) How many cars are there in the car park of an average

(2) what is the probability of a newly arriving customer finding the car park full and leaving to park his car elsewhere

Solution

one car park → Single channel

Arrival of cars → 5 (finite)

Model → $(M/M/1):(N/FCFS)$

Mean arrival rate $\lambda = \frac{10}{60} = \frac{1}{6}$ per minutes

Mean service time $\frac{1}{\mu} = 2$ minutes

Mean Service rate $\mu = \frac{1}{2}$ minutes

$= \frac{1}{2} \times 60$ per hour

$= 30$ per hour

Capacity of the system $N = 5$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{30} = 0.333.$$

$$= \frac{1}{3}$$

$$P_0 = \frac{1-\rho}{1-(\rho)^{N+1}}$$

$$= \frac{1-1/3}{1-(1/3)^6} = \frac{1-1.3}{0.9986} = \frac{0.6666}{0.9986}$$

$$= 0.6675$$

$$= 0.667$$

(1) Average car in the park

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-(\rho)^{N+1}}$$

$$= \frac{1/3}{1-1/3} - \frac{(5+1)(1/3)^{5+1}}{1-(1/3)^{5+1}}$$

$$= \frac{1/3}{2/3} - \frac{6(1/3)^6}{1-(1/3)^6}$$

$$= \frac{1}{2} - \frac{6\left(\frac{1}{729}\right)}{1-\frac{1}{729}}$$

$$= 0.492$$

(2) P(system is full) = P(5 cars in the syst

$$= P(N=5)$$

$$= P_5$$

$$= \rho^N P_0$$

$$= (1/3)^5 (0.668)$$

$$= 0.0027$$

(32)

⑥ Trains arrive at the yard every 15 minutes $= 1/15$ and the service time is 33 minutes. If the line capacity at the yard is limited to 4 trains, find

(1) The probability that the yard is empty

(2) The average number of trains in the system

Solution

one yard → single channel

Arrival of trains → 4 (finite)

Model → (M/M/1) : (N/FCFS)

Mean arrival rate $\lambda = \frac{1}{15}$ per min

Mean service time $\frac{1}{\mu} = 33$ min

Mean Service rate, $\mu = \frac{1}{33}$ per min

Capacity of the system $N = 4$

$$\rho = \frac{\lambda}{\mu}$$

$$= \frac{33}{15} = 2.2$$

(1) Probability (yard is empty)

$$P_0 = \frac{1-\rho}{1-(\rho)^{N+1}}$$

$$= \frac{1-2.2}{1-(2.2)^5}$$

$$= \frac{-1.2}{1-51.5}$$

$$= \frac{-1.2}{-50.5}$$

$$= 0.0237$$

(2) Average number of trains in the system

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-(\rho)^{N+1}}$$

$$= \frac{2.2}{1-2.2} - \frac{(4+1)(2.2)^{4+1}}{1-(2.2)^{4+1}}$$

$$= 3.26$$

(7) In a railway marshalling yard, goods trains arrive at the rate of 30 trains per day. Assume that the interarrival time follows an exponential distribution and the service time is also to be assumed an exponential with mean of 36 minutes calculate

(1) The probability that the yard is empty

(2) The average queue length, assuming the line capacity of the yard is 9 trains

Solution one yard → single channel

Capacity of the yard → 9 (finite)

Model → $(M/M/1):(N/FCFS)$

Mean arrival rate $\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$ per m

Mean service time $\frac{1}{\mu} = 36$ minutes

Capacity of the system $N = 9$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{48}}{\frac{1}{36}} = \frac{1}{48} \times 36 = 3/4$$

$$\rho = 0.75$$

① P(The yard is empty)

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-0.75}{1-(0.75)^{10}}$$

$$= \frac{0.25}{0.94}$$

$$= 0.27$$

② Average queue length

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$= \frac{0.75}{1-0.75} - \frac{(9+1)(0.75)^{9+1}}{1-(0.75)^{9+1}}$$

$$= \frac{0.75}{0.25} - \frac{10(0.0563)}{1-0.0563}$$

$$= 2.4$$

---

⑧ In a single server queueing system with Poisson input and exponential service time, If the mean arrival rate is 3 calling units $= \frac{1}{0.025}$ per hour, the expected service time is 0.25 hr and the maximum possible number of calling unit in the system is ^{N}2 then [N/D-2017]

① find $P_n (n \geq 0)$

② Average number of calling units in the system and in the queue

③ Average waiting time in the system and in the queue

Solution one server → single server [1]
calling units → finite capacity [k] [N]
Here $N=2$

∴ Hence this problem comes under the model $(M/M/1):(N/FIFO)$

Given Mean arrival rate $(\lambda) = 3$ per hour
Mean service time $= 0.25$ hour
Mean service rate $(\mu) = \frac{1}{0.25}$ per hour
$= 4$ per hour

$$\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^3} = \frac{0.25}{0.578}$$

$$= 0.433 \quad [0.43235]$$

$$\lambda' = \mu(1-P_0) = 4(1-0.433) = 2.27 \quad (2.268)$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$= \frac{0.75}{1-0.75} - \frac{(2+1)(0.75)^{2+1}}{1-(0.75)^{2+1}} = \frac{1.2656}{1-4218}$$

$$= 3 - 2.189 = 0.81 \quad = 3-2.189 = 0.811$$

$$L_q = L_s - \frac{\lambda'}{\mu} = 0.81 - \frac{2.27}{4} = 0.2425$$

$$W_s = \frac{1}{\lambda'} L_s$$

$$= \left(\frac{1}{2.27}\right)(0.81) = 0.3568$$

$= 0.36h = 0.36 \times 60$ min

$= 21.6$ min $= 21.41$

$W_q = \frac{1}{\lambda'} L_q$

$= \left(\frac{1}{2.27}\right)(0.2425)$

$= 0.11h = (0.11)(60)$ mn

$= 6.6$ mn

$P_n = \rho^n P_0 = (0.75)^n (0.4333) = (0.75)^2 (0.4333) = (0.5625)(0.4333) = 0.2437.$

The average number of calling units in the system $= L_s = 0.81$

The average number of calling units in the queue $= L_q = 0.2425$

Average waiting time in the system

$W_s = 21.6$ mn

Average waiting time in the queue

$W_q = 6.6$ min

Model IV [M/M/C]:[K/FCFS] (or) [K/FIFO]

(1) $\rho = \frac{\lambda}{s\mu}$

$$P_0 = \left[\left(\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}\left[1+\rho+\rho^2+\cdots+\rho^{k-s}\right)\right]^{-1}$$

(2) $L_s = \sum_{n=0}^{k} n P_n$

$L_q = L_s - \frac{\lambda'}{\mu}$.

$W_s = \frac{1}{\lambda'} L_s$

$W_q = \frac{1}{\lambda'} L_q$

$$\lambda' = \mu\left[s - \sum_{n=0}^{s-1} (s-n) P_n\right]$$

$s=2 \Rightarrow \frac{\lambda'}{\mu} = 2 - (2P_0 + P_1)$

$s=3 \Rightarrow \lambda'/\mu = 3 - (3P_0 + 2P_1 + P_2)$

$s=4 \Rightarrow \lambda'/\mu = 4 - (4P_0 + 3P_1 + 2P_2 + P_3)$

(M/M/c) : (k/FIFO) Model

(1) A car servicing station has 2 bays where service can be offered simultaneously. Because of space limitation, only 4 cars are accepted for servicing. The arrivals pattern is poisson with 12 car per day. The service time in both the bays is exponentially distributed with $\mu = 8$ cars per day. Find (1) The average number of cars in the service distribution

(2) The average number of cars waiting for service

(3) The average time a car spends in the system

Solution Given 2 bays → multiple server

4 cars → finite capacity

This problems under (M/M/c) : (k/FIFO)

Given $\lambda = 12$. $\mu = 8$ $s = 2$ $k = 4$

$$\rho = \frac{\lambda}{s\mu} = \frac{12}{2(8)} = 0.75$$

$$p_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{\lfloor n} + \frac{(s\rho)^s}{\lfloor s}(1+\rho+\rho^2+\dots+\rho^{k-s})\right]^{-1}$$

For $s = 2$

$$p_0 = \left[1 + \frac{(2\rho)^1}{\lfloor 1} + \frac{(2\rho)^2}{\lfloor 2}(1+\rho+\rho^2+\dots+\rho^{k-2})\right]^{-1} \quad \text{--- (1)}$$

$$\frac{\lambda'}{\mu} = 2 - (2p_0 + p_1) \quad \text{--- (2)}$$

For $k = 4$,

$$p_0 = \left[1 + \frac{(2\rho)^1}{\lfloor 1} + \frac{(2\rho)^2}{\lfloor 2}(1+\rho+\rho^2)\right]^{-1}$$

$$= \left[1 + (2\times0.75) + \left[\frac{(2\times0.75)^2}{\lfloor 2}\right](1+0.75+(0.75)^2)\right]^{-1}$$

$$= 0.196$$

$p_1 = \frac{(2\rho)^1}{\lfloor 1} p_0 = 2\rho p_0 = 2(0.75)(0.196)$
$= 0.294$

$p_2 = \frac{(2\rho)^2}{\lfloor 2} p_0 = 2\rho^2 p_0 = (2)(0.75)^2(0.196)$
$= 0.2205$

$p_3 = p_2 \rho = (0.2205)(0.75) = 0.1654$

$p_4 = p_3 \rho = (0.1654)(0.75) = 0.1241$

(2) $\Rightarrow \frac{\lambda'}{\mu} = 2 - (2p_0 + p_1)$
$= 2 - [2(0.191) + 0.294] = 1.314$

$\lambda' = (1.314)\mu = (1.314)(8) = 10.512$

$\frac{1}{\lambda'} = 0.0951$

$L_s = 1p_1 + 2p_2 + 3p_3 + \cdots + kp_k$
$= 1p_1 + 2p_2 + 3p_3 + 4p_4 \; [\because k = 4]$
$= 0.294 + 2(0.2205) + 3(0.1654) + 4(0.124)$
$= 1.7276$

$L_q = L_s - \frac{\lambda'}{\mu}$
$= 1.7276 - 1.314 = 0.4136$

$W_s = \frac{1}{\lambda'} L_s$
$= (0.0951)(1.7276) = 0.1643$

$W_q = \frac{1}{\lambda'} L_q = (0.0951)(0.4136) = 0.0393$

(1) The average number of cars in the service station $= L_s = 1.73$

(2) The average number of cars waiting for service $L_q = 0.41$

(3) The average time a car spends in the system $W_s = 0.1643$ days
$= 0.1643 \times 24$ hrs
$= 3.94$ hours

(39)

(2) At a port there are 6 unloading berths and 4 unloading crews. When all the berths are full arriving ships are diverted to an overflow facility 20 kms down the river. Tankers arrive according to a poisson process with a mean of 1 every 2 hours. It takes for an unloading crew, on the average, 10 hours to unload a tanker, the unloading time following an exponential distribution

(a) How many tankers are at the port on the average

(b) How long tanker spend at the port on the average

(c) what is the average arrival rate at the overflow facility

<u>Solution</u> Given 4 unloading crews → multiple server

6 unloading berths → finite capacity

Given $\lambda = \frac{1}{2}, \mu = \frac{1}{10}, S=4, k=6$

$$\rho = \frac{\lambda}{S\mu} = \frac{1/2}{(4)\left(\frac{1}{10}\right)} = \frac{5}{4} = 1.25$$

$$P_0 = \left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{\lfloor n} + \frac{(S\rho)^S}{\lfloor S}\left(1+\rho+\rho^2+\cdots+\rho^{k-S}\right)\right]^{-1}$$

For $S=4$

$$P_0 = \left[1 + \frac{(4\rho)^1}{\lfloor 1} + \frac{(4\rho)^2}{\lfloor 2} + \frac{(4\rho)^3}{\lfloor 3} + \frac{(4\rho)^4}{\lfloor 4}\left(1+\rho+\rho^2+\cdots+\rho^{k}\right)\right]^{-1} \quad (1)$$

$$\frac{\lambda'}{\mu} = 4 - (4P_0 + 3P_1 + 2P_2 + P_3) \quad (2)$$

For $k=6$

$$(1) \Rightarrow P_0 = \left[1 + \frac{(4\rho)^1}{\lfloor 1} + \frac{(4\rho)^2}{\lfloor 2} + \frac{(4\rho)^3}{\lfloor 3} + \frac{(4\rho)^4}{\lfloor 4}\left(1+\rho+\rho^2\right)\right]^{-1}$$

$$= \left[1 + (4\times 1.25) + \left[\frac{(4\times 1.25)^2}{2!}\right] + \left[\frac{(4\times 1.25)^3}{\lfloor 3}\right.\right.$$

$$+\left[\frac{(4\times 1.25)^4}{\lfloor 4}\left(1+1.25+1.25^2\right)\right]^{-1}$$

$$= 0.0072$$

$$P_1 = \frac{(4\rho)^1}{\lfloor 1} P_0 = 4\rho P_0 = (4)(1.25)(0.0072) = 0.036$$

$$P_2 = \frac{(4\rho)^2}{\lfloor 2} P_0 = 8\rho^2 P_0 = (8)(1.25)^2(0.0072) = 0.09$$

$$P_3 = \frac{(4\rho)^3}{\lfloor 3} P_0 = \frac{64}{6}\rho^3 P_0 = \left(\frac{64}{6}\right)(1.25)^3(0.0072) = 0.15$$

$$P_4 = \frac{(4\rho)^4}{\lfloor 4} P_0 = \frac{256}{24}\rho^4 P_0 = \left(\frac{256}{24}\right)(1.25)^4(0.0072)$$

$$= 0.1875$$

$$P_5 = P_4\rho = (0.1875)(1.25)$$

$$= 0.2344$$

$$P_6 = P_5\rho = (0.2344)(1.25) = 0.293$$

$$\Rightarrow \frac{\lambda'}{\mu} = 4 - (4P_0 + 3P_1 + 2P_2 + P_3)$$

$$= 4 - \left[4(0.0072) + (3)(0.036) + 2(0.09) + 0.15\right]$$

$$= 3.5332$$

$$\lambda' = (3.5332)\mu$$

$$= (3.5332)\left(\frac{1}{10}\right)$$

$$= 0.3533$$

$$\frac{1}{\lambda'} = 2.8305$$

$$L_s = 1P_1 + 2P_2 + 3P_3 + \cdots + kP_k$$

$$= 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 + 6P_6$$

$$= 0.036 + 2(0.09) + 3(0.15) + 4(0.1875) + (5)(0.2344) + (6)(0.293)$$

$$= 4.346$$

$$L_q = L_s - \frac{\lambda'}{\mu} = 4.346 - 3.5332 = 0.8128$$

$$W_s = \frac{1}{\lambda'} L_s = (2.8305)(4.346) = 12.3014$$

$$W_q = \frac{1}{\lambda'} L_q = (2.8305)(0.8128) = 2.3006$$

Number of tankers at the port on the average $= L_q = 0.8128$

Average waiting time by a tanker at the port $W_s = 12.3014$

Average arrival at the over flow facility

$= [\text{average arrival at the port}] \times P_6$

$= \left(\frac{1}{2}\right)(0.293)$

$= 0.1465$ per hour

Multiserver queue [Model II]

Model $[M/M/s]:[\infty/FCFS]$ (or) $[M/M/c]$ $[\infty$ (or) $/FIFO]$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!\left(1-\lambda/\mu s\right)}\left(\frac{\lambda}{\mu}\right)^s\right]^{-1}$$

(or)

$$* \quad P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{\lfloor n} + \frac{(s\rho)^s}{s!}(1-\rho)^{-1}\right]^{-1}$$

$$* \quad L_s = \frac{\rho}{(1-\rho)^2}\frac{(s\rho)^s}{s!}P_0 + s\rho$$

$$* \quad \rho = \frac{\lambda}{s\mu}$$

$$* \quad L_q = L_s - \lambda/\mu$$

$$* \quad W_s = \frac{1}{\lambda}L_s$$

$$* \quad W_q = \frac{1}{\lambda}L_q$$

* The probability that an arrival has to wait = the probability that there are s (or) more customers in the system

$$P(W_s > 0) = P(N \geq s) = \frac{(\lambda/\mu)^s}{s!\left(1-\lambda/\mu s\right)}P_0$$

$$= \frac{(\lambda/\mu)^s P_0}{s!(1-\rho)}$$

* The mean waiting time in the queue for those who actually wait

$$E\left[W_q / W_s > 0\right] = \frac{1}{\mu s - \lambda}$$

UNIT-3 :-

MODEL 1 = (M/M/1) : (∞/FCFS)

MODEL 2 = (M/M/S) : (∞/FCFS)

MODEL 3 = (M/M/1) : (K/FCFS)

MODEL 4 = (M/M/1) : (K/FCFS)

MODEL I :	MODEL II :	MODEL III	MODEL IV.
1. $\rho = \frac{\lambda}{\mu}$	$\rho = \frac{\lambda}{s\mu}$	$\rho = \frac{\lambda}{\mu}$	$\rho = \lambda / s\mu$
2. $P_0 = 1-\rho$	$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}(1-\rho)^{-1}\right]^{-1}$	$P_0 = \frac{1-\rho}{(1-\rho)^{N+1}}, \lambda \neq \mu$	$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}[1+\rho+\rho^2+\rho^{k-s}]\right]^{-1}$
3. $L_s = \frac{\rho}{1-\rho}$	$L_s = \frac{\rho}{(1-\rho)^2} \frac{(s\rho)^s}{s!} P_0 + s\rho$	$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$	$L_s = \sum_{n=0}^{K} n P_n$
4. $L_q = L_s - \frac{\lambda}{\mu}$	$L_q = L_s - \frac{\lambda}{\mu}$	$L_q = L_s - \lambda'/\mu$	$L_q = L_s - \lambda'/\mu$
5. $W_s = \frac{1}{\lambda} L_s$	$W_s = \frac{1}{\lambda} L_s$	$W_s = \frac{1}{\lambda'} L_s$	$W_s = \frac{1}{\lambda'} L_s$
6. $W_q = \frac{1}{\lambda} W_q$	$W_q = \frac{1}{\lambda} L_q$	$W_q = \frac{1}{\lambda'} L_q$	$W_q = \frac{1}{\lambda'} L_q$, $\lambda' = \mu(1-P_0)$
7. $L_w = \frac{1}{1-\rho}$		$\lambda' = \mu(1-P_0)$	
8. $P[w>k] = e^{-(\mu-\lambda)k}$			
9. $P(N>k) = \rho^{k+1}$			
10. $P(N \geq k) = \rho^k$			

① A petrol pump station has 4 pumps. The service times follows the exponential distribution with a mean of 6 min. and cars arrive for service in a Poisson process at the rate of 30 cars per hour.

① What is the probability that an arrival would have to wait in line

② Find the average waiting time in the queue, average time spent in the system and the average number of cars in the system

③ For what % of time would a pump be idle on an average

<u>Solution</u> Given petrol pumps → Multiple server [s]

cars → Infinite capacity [∞]

$s = 4$

The given problem is $[M/M/S]:[\infty/FIFO]$

Mean arrival rate $(\lambda) = 30$ per hour

Mean service time = 6 minutes

Mean service rate $(\mu) = \frac{1}{6}$ per minutes

$= \frac{1}{6} \times 60$ per hour

$= 10$ per hour

$s = 4, \lambda = 30, \mu = 10$

[Formula] $\rho = \frac{\lambda}{s\mu} = \frac{30}{(4)(10)} = 0.75$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}(1-\rho)^{-1}\right]^{-1}$$

For $s = 4$

$$P_0 = \left[1 + \frac{(4\rho)^1}{\lfloor 1} + \frac{(4\rho)^2}{\lfloor 2} + \frac{(4\rho)^3}{\lfloor 3} + \frac{(4\rho)^4}{\lfloor 4}(1-\rho)^{-1}\right]^{-1}$$

$= 0.0377$ [Here $\rho = 0.75$ use your calculator]

① P[an arrival has to wait]

[formula
$$P[N \geq S] = \frac{(\lambda/\mu)^s}{S!\,(1-\rho)} p_0$$]

$$P(N \geq 4) = \frac{3^4}{4!\,(0.25)} (0.0377)$$

$$= 0.509$$

② The average number of cars in the system

$$L_s = \frac{\rho}{(1-\rho)^2} \frac{(s\rho)^s}{s!} p_0 + s\rho \quad \text{[formula]}$$

$$= \frac{\rho}{(1-\rho)^2} \frac{(4\rho)^4}{4!} p_0 + 4\rho$$

$$= 4.53 \text{ cars} \quad [\text{Here } \rho = 0.75]$$

The average time spent in the system

$$W_s = \frac{1}{\lambda} L_s = \frac{1}{30}(4.53) = 0.151\,h$$

$$= (0.151)(60)\,min$$

$$= 9.06\,min$$

The average waiting time in the queue

$$W_q = \frac{1}{\lambda} L_q \qquad \left[\because L_q = L_s - \lambda/\mu\right.$$

$$= 4.53 - 3 = 1.53$$

$$= \frac{1}{30}(1.53)$$

$$= 0.051\,h$$

$$= (0.051)(60)\,min = 3.06\,min$$

③ The fraction of time when the pumps are busy = traffic intensity

$$= \frac{\lambda}{\mu s} = \frac{3}{4}$$

∴ The fraction of time when the pumps are idle $= 1 - \frac{3}{4}$

∴ The required percentage = 25%

② There are 3 typist in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour

① What fraction of the time all the typists will be busy

② What is the average number of letters waiting to be typed

③ What is the average time a letter has to spend for waiting and for being typed

④ What is the probability that a letter will take longer than 20 min waiting to be typed and being typed [A/M - 2017] [A/N/D - 2016]

Solution Given Typist → multiple server [S]

Letters → infinite capacity [∞]

The given problem is (M/M/S) : (∞/FIFO) Model

Given Mean arrival rate $(\lambda) = 15$ per hour

Mean service rate $(\mu) = 6$ per hour

$S = 3$

$S = 3, \lambda = 15, \mu = 6$

$$\rho = \frac{\lambda}{S\mu} = \frac{15}{(3)(6)} = 0.83 \Rightarrow \boxed{\rho = 0.83}$$

$$P_0 = \left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!} (1-\rho)^{-1}\right]^{-1} \text{ [formula]}$$

For S = 3
$$P_0 = \left[1 + \frac{(3\rho)^1}{1!} + \frac{(3\rho)^2}{2!} + \frac{(3\rho)^3}{3!} (1-\rho)^{-1}\right]^{-1}$$

[Here $\rho = 0.83$ use calculator]

$$\boxed{P_0 = 0.046}$$

$$L_s = \frac{\rho}{(1-\rho)^2} \frac{(S\rho)^S}{S!} P_0 + S\rho \quad \left[\text{Put } P_0 = 0.046,\ \rho = 0.83\right]$$

$$= \frac{\rho}{(1-\rho)^2} \frac{(3\rho)^3}{3!} P_0 + 3\rho = 6$$

$L_q = L_s - \lambda/\mu = 6 - 2.5 = 3.5$

$W_s = \frac{1}{\lambda} L_s = \left(\frac{1}{15}\right)(6) = 0.4\,h$

$= (0.4)(60)\,min = 24\,min$

$W_q = \frac{1}{\lambda} L_q = \left(\frac{1}{15}\right)(3.5) = 0.2333$

(i) P[All the typist are busy] = $P[N \geq 3]$

[formula $P[N \geq s] = \frac{(\lambda/\mu)^s P_0}{s!\,(1-\rho)}$]

$= \frac{(2.5)^3 (0.046)}{3!\,(1-0.83)} = 0.70$

Hence, the fraction of the time all the typist will be busy = 0.70

(ii) The average number of letters waiting to be served (or typed)

$L_q = L_s - \lambda/\mu = 3.5$

(iii) The average waiting time a letter has to spend for waiting and for being typed

$W_s = \frac{1}{\lambda} L_s = 24\,min$

(iv) $P(W > t) = e^{-\mu t}\left\{1 + \frac{(\lambda/\mu)^s\left[1 - e^{-\mu t(s-1-\lambda/\mu)}\right]}{s!\,(1-\lambda/\mu s)(s-1-\lambda/\mu)} P_0\right\}$

[assumed formula] [20 min = 1/3 h]

$\therefore P[W > 1/3] = e^{-6 \times 1/3}\left\{1 + \frac{(2.5)^3\left[1 - e^{-2 \times (-0.5)}\right]}{6\left[1 - \frac{2.5}{3}\right][-}\right.$

$= 0.4616$

(417)

③ A supermarket has 2 girls attending to sales at the counters. If the service time for each customer is exponential with mean 4 min and if people arrive in poisson fashion at the rate of 10 per hour

① what is the probability that a customer has to wait for service

② What is the expected % of idle time for each girl

③ If the customer has to wait in the queue, what is the expected length of his waiting time

Solution

girls → multiple server [S]
people → infinite capacity [∞]

∴ The given problem is (M/M/S) : (∞/FIFO) model

Given $S=2$

Mean arrival rate $(\lambda) = 10$ per hour

$= 10 \times \frac{1}{60}$ per minute

$= \frac{1}{6}$ per minute

Mean service time

$= 4$ minutes

Mean service rate $(\mu) = \frac{1}{4}$ per minute

$S = 2, \lambda = 1/6, \mu = 1/4$

$$\rho = \frac{\lambda}{S\mu} = \frac{1/6}{(2)(1/4)} = \frac{2}{6} = 0.33 \Rightarrow \boxed{\rho = 0.33}$$

$$P_0 = \left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!}(1-\rho)^{-1}\right]^{-1}$$

For $S=2$

$$P_0 = \left[1 + \frac{(2\rho)^1}{\lfloor 1} + \frac{(2\rho)^2}{\lfloor 2}(1-\rho)^{-1}\right]^{-1} = 0.5 \Rightarrow \boxed{P_0 = 0.5}$$

Here $\rho = 0.33$, use calculator

① P[a customer has to wait] = $P[N \geq 2]$

$$P[N \geq 2] = \frac{1}{2!}\frac{(0.67)^2}{(0.67)}(0.5)$$

$= 0.168$

Formula, $P[N \geq S] = \frac{1}{S!}\frac{(\lambda/\mu)^S}{(1-\rho)}P_0$

(ii) The fraction of time when the girls are busy

$$= \frac{\lambda}{\mu s} = \frac{1}{3}$$

∴ The fraction of time when the girls are idle

$$= 1 - 1/3 = 2/3$$

∴ The expected % of idle time for each girl

$$= \frac{2}{3} \times 100 = 67$$

(iii) $E\left[W_q / W_s > 0\right] = \frac{1}{\mu s - \lambda}$ [formula]

$$= \frac{1}{\frac{1}{4} \times 2 - 1/6} = \frac{1}{\frac{1}{2} - \frac{1}{6}}$$

$$= \frac{1}{\frac{6-2}{12}} = \frac{1}{4/12} = \frac{12}{4}$$

$$= 3 \text{ min}$$

(4) A telephone exchange has 2 long distance operators. The telephone company finds that during the peak lo[ad] long distance calls arrive in a poisson fashion at a[n] average rate of 15 per hour. The length of service on calls is approximately exponentially distributed with Mean length 5 min

(a) What is the probability that a subscriber will have wait for his long distance call during the peak of the day

(b) If the subscribers will wait and are serviced in What is the expected waiting time?

Solution

operators → multiple server [S]

calls → infinite capacity [∞]

∴ The given problem is $(M/M/S):(\infty/FIFO)$ model

Given $S=2$

Mean arrival rate $(\lambda) = 15$ per hour

$= 15 \times \frac{1}{60}$ per minute

$= \frac{1}{4}$ per minute

Mean service time $= 5$ minutes

Mean Service rate $(\mu) = \frac{1}{5}$ per minutes

$S=2, \quad \lambda = 1/4, \quad \mu = 1/5$

$\rho = \frac{\lambda}{s\mu} = \frac{1/4}{(2)(1/5)} = 5/8 = 0.625 \Rightarrow \boxed{\rho = 0.625}$

$$P_0 = \left[\left(\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!}\right) + \frac{(s\rho)^s}{s!}(1-\rho)^{-1}\right]^{-1}$$

<u>For $s=2$</u> $\quad P_0 = \left[1 + \frac{(2\rho)^1}{\lfloor 1} + \frac{(2\rho)^2}{\lfloor 2}(1-\rho)^{-1}\right]^{-1}$

$\boxed{P_0 = 0.231}$ [Here $\rho = 0.625$ use your calculator]

$$L_s = \frac{\rho}{(1-\rho)^2}\frac{(s\rho)^s}{s!}P_0 + s\rho$$

$$= \frac{\rho}{(1-\rho)^2}\frac{(2\rho)^2}{2!}P_0 + 2\rho \quad \text{here } P = 0.231$$

$\boxed{L_s = 2.052}$

$L_q = L_s - \lambda/\mu$

$= 2.052 - 1.25 = 0.802$

$W_s = \frac{1}{\lambda}L_s = \frac{1}{(1/4)}(2.052)$

$= 8.208$

$W_q = \frac{1}{\lambda}L_q = \frac{1}{(1/4)}(0.802) = (4)(0.802)$

$= 3.208$ min

(a) P[a customer waits] = P[N ≥ 2]

Formula $P[N \geq s] = \dfrac{(\lambda/\mu)^s p_0}{s!\,(1-\rho)}$

$$= \frac{(1.25)^2 (0.231)}{(2!)(0.375)}$$

$= 0.48$

(b) The expected waiting time

$= W_q = 3.2$ min

PART-A

(1) What are the basic chacteristic of a queueing system?

(1) Arrival pattern of customers
(2) Service pattern of customers
(3) Queue discipline and
(4) System capacity

(2) What do you mean by transient state and steady state queueing system?

Steady State:
If the chacteristics of a queueing system are independent of time

Transient State:
If the chacteristic of a queueing system are dependent of time

The necessary condition for a system to be steady state in $\rho < 1$ i.e., $\lambda < \mu$

(3) What do the letters in the symbolic representation of a queueing model represent? (a/b/c):(d/e) of a queueing model is specified and usually a queueing model represented symbolically in the form (a/b/c):(d/e)

a → the type of distribution of the number of arrivals per unit time

b → The type of distribution of the service time c → The number of servers

d → the capacity of the system, the maximum queue size

e → The queue discipline

④ Discuss the terms

Balking :
A customers who leaves the queue because the queue is too long and he was no time to wait or has no sufficient waiting space

Reneging :
This occurs when a waiting customer leaves the queue due to impatience

Jockeying : customers may jockey from one waiting line to another. This is most common in supermarket.

⑤ Define little's formula,
(or)
Write the relations among Ls, Lq, Ws and Wq

$W_s = \frac{1}{\mu - \lambda}$

$W_q = W_s - 1/\mu$

$L_s = \lambda W_s$

$L_q = \lambda W_q$

(6) What is the probability that a customer has to wait more than 15 minutes to get his service completed is $(M/M/1):(\infty/FIFO)$ queue system if $\lambda = 6$ per hour and $\mu = 10$ per hour?

Solution. The probability that the waiting time of customer in the system exceeds

$$t = e^{-(\mu-\lambda)t}$$

Given $\lambda = 6$, $\mu = 10$ and $t = 15 \text{ min} = \frac{1}{4}$ hr

$\therefore$ The required probability $= e^{-(10-6)\frac{1}{4}}$

$$= e^{-1}$$

$$= \frac{1}{e} = 0.3679$$

(7) If $\lambda = 4$/hrs and $\mu = 12$/hrs in an $(M/M/1)$ $(4/FIFO)$ queueing system, find the probability that there is no customers in the system

Solution: Given $\lambda = 4$/hrs

$$\mu = 12/\text{hrs}$$

$$\rho = \lambda/\mu = \frac{4}{12} = \frac{1}{3}$$

$$N = 4$$

We know that, the probability that there is no customers in the system is

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-\frac{1}{3}}{1-\left(\frac{1}{3}\right)^{4+1}} = \frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^5} = \frac{81}{121}$$

(8) If $\lambda = 3/hr$, $\mu = 4/hr$ and maximum capacity N=7 in a (M/M/1):(N/FIFO) system, find the average number of customers in the system

Solution Given $\lambda = 3/hr$

$\mu = 4/hr$

$\rho = \lambda/\mu = 3/4$

$N = 7$

We know that, average number of customers in the system is

$$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(N+1)(\lambda/\mu)^{N+1}}{1 - (\lambda/\mu)^{N+1}}$$

$$= \frac{3}{4-3} - \frac{(7+1)(3/4)^8}{1 - (3/4)^8}$$

$$= \frac{3}{1} - \frac{8(3/4)^8}{1 - (3/4)^8}$$

$$= 3 - 0.8898$$

$L_s \simeq 2.11$ customers

UNIT-5

UNIT :5 ADVANCED QUEUEING MODELS (1)

Pollaczek - Khintchine formula :- (S) V.V.I

(OR)

Derive the expected steady state system size for the signal server queues with poisson input and general services.

Let $n' = n-1 + \delta + k$ - - - (1)

$n \rightarrow$ number of customers in the system at time 't'.

$n' \rightarrow$ number of customers in the system at time 't+T'

$T \rightarrow$ random service time

$K \rightarrow$ number of arrivals during the service time

$$\delta = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

$$(1) \Rightarrow n' = \begin{cases} k & \text{if } n=0 \\ (n-1)+k & \text{if } n>0 \end{cases}$$

$\delta^2 = \delta$ . . . (2) [for values of $\delta = 0$ and $\delta = 1$]

$n\delta = 0$ - - - (3) [$\because$ $n=0$, $\delta=1 \Rightarrow n\delta = 0$
$n>0$, $\delta = 0 \Rightarrow n\delta = 0$]

(1) $= n' - n = -1 + \delta + K$

$E[n'] - E[n] = -1 + E[\delta] + E[K]$ [delta] $[\because E[1] = 1]$ &

$0 = -1 + E[\delta] + E[K]$

Insteady state

$\because E[n'] = E[n]$

& $E[n'^2] = E[n^2]$ ---- (5)

$E(\delta) = 1 - E[K]$ ------ (4)

$(1) \Rightarrow n' = n - 1 + \delta + K$
a b c d

$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad$
$2bc + 2bd + 2cd$

$n'^2 = n^2 + 1 + \delta^2 + K^2 - 2n + 2n\delta + 2nK - 2\delta$
$+ 2\delta K$

$= n^2 + 1 + \delta + K^2 - 2n + 0 + 2nK - 2\delta - 2K + 2\delta$

by (2) & (3)

$n'^2 - n^2 + 2n - 2nK = 1 - \delta + K^2 - 2K + 2\delta K$

$n'^2 - n^2 + 2n(1-K) = 1 - \delta + K^2 - 2K + 2\delta K$

$E[n'^2] - E[n^2] + 2E(n)[1 - E(K)] =$
$1 - E[\delta] + E[K^2] - 2E[K] +$
$2E[\delta]E[K]$

$$2E[n][1-E(K)] = E[K] + E[K^2] - 2E[K] + 2[1-E(K)]E[K] \quad \text{by (4) \& (5)}$$

$[1-E(B) = E(K)]$.

$$= E[K] + E[K^2] - 2E[K] + 2E[K] - 2E[K]^2$$

$$= E[K^2] - E[K] + 2[1-E(K)]E[K]$$

$$E[n] = \frac{E[K^2] - E[K]}{2[1-E(K)]} + E[K] \quad \text{- - - -(6)}$$

K arrivals in T follows poisson process with parameter λ

In poisson process

$$E[X(t)] = \lambda t$$

$$E[X^2(t)] = (\lambda t)^2 + \lambda t$$

$$V(T) = E[T^2] - [E(T)]^2$$

$$\Rightarrow E[T^2] = V(T) + [E(T)]^2$$

$$E(K) = \lambda E[T]$$

$$E[K^2] = \lambda^2 E[T^2] + \lambda E[T]$$

$$E[K^2] - E[K] = \lambda^2 E[T^2]$$

$$= \lambda^2 [V(T) + [E(T)]^2]$$

$$= \lambda^2 V(T) + \lambda^2 [E(T)]^2$$

$$(6) \Rightarrow E[n] = \frac{\lambda^2 V(T) + \lambda^2 [E(T)]^2}{2[1-\lambda E(T)]} + \lambda E[T] \text{ - - (7)}$$

We know that,

$$\mu = \frac{1}{E(T)}, \quad E(T) = \frac{1}{\mu},$$

$$\lambda E(T) = \frac{\lambda}{\mu} = \rho, \quad V(T) = \sigma^2$$

$$(7) \Rightarrow E(n) = L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho$$

i.e.,

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$W_s = \frac{1}{\lambda} L_s$$

$$W_q = \frac{1}{\lambda} L_q$$

Example → Book pg no: 5.2

ficiency of a queuing system:

In order to measure the efficiency of a queuing system, a ratio may be defined as,

$$\frac{W_q}{E(t)} = \frac{\text{average waiting in queue}}{\text{average service time}} = \frac{\text{useless time}}{\text{useful time}}$$

Thus smaller the ratio better will be the system.

M/D/1 Queue

M -> arrival time follows poisson distribution

D -> service time follows constant distribution

1 -> single server model

Here, the service time is constant

Hence, $Var(T) = 0$, i.e., $\sigma^2 = 0$

Hence, from P-K formula, we get

$$L_s = \rho + \frac{\rho^2}{2(1-\rho)}, \quad \text{where } \rho = \frac{\lambda}{\mu}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$W_s = \frac{1}{\lambda} L_s$$

$$W_q = \frac{1}{\lambda} L_q$$

Example: State P-K formula for the average number in the system in a M/G/1 model and hence, derive the same when the service time is constant with mean $\frac{1}{\mu}$.

2 mark

Sol

P-K formula

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

Given: Service time is constant

i.e., $\sigma^2 = 0$, $\therefore L_s = \rho + \frac{\rho^2}{2(1-\rho)}$

M/Ek/1 queue

M → arrival time follows Poisson distribution

Ek → Service time follows Erlang distribution with K phases

1 → single server model

Here, $\mu = \frac{1}{km}$, $\sigma^2 = \frac{1}{k\mu^2}$, $\rho = \frac{\lambda}{\mu}$

Hence, from P-K formula, we get

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left[\frac{1}{k\mu^2}\right] + \left(\frac{\lambda}{\mu}\right)^2}{2\left[1+\frac{\lambda}{\mu}\right]}$$

$$= \frac{\lambda}{\mu} + \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$L_q = L_s = \frac{\lambda}{\mu}$$

$$W_s = \frac{1}{\lambda} L_s$$

$$W_q = \frac{1}{\lambda} L_q$$

$$\text{Mode} = \frac{k-1}{\mu k}$$

Mode → Most probable time spent in the service

Note:

When k = 1 in the M/EK/1 model, the Erlang service time reduces to exponential distribution. or, in general M/G/1 queue, when service time distribution is exponential, we get (M/M/1) queue.

Here, $L_s = \frac{\rho}{1-\rho}$, $L_q = L_s - \frac{\lambda}{\mu}$, $W_s = \frac{1}{\lambda} L_s$,

$$W_q = \frac{1}{\lambda} L_q$$

WORKED EXAMPLES

TYPE I : problems based on $(M/G/1) : (\infty/FIFO)$ m

Example 5.1.1

In a heavy machine shop, the overhead crane is 75% utilised. Time study observati gave the average slinging time as 10.5 mi with a standard deviation of 8.8 minutes.

(a) What is the average calling rate for the services of the crane? and

(b) What is the average delay in getting

(c) If the average service time is cut to minutes, with a standard deviation of 6.(minutes, how many reduction will occur, average, in the delay of getting served,

sol :-

This is a $(M/G/1) : (\infty/FIFO)$ process

Given:

$$\text{Utilisation rate} = 75\% = \frac{75}{100} = \frac{3}{4} = 0.75$$

$$\rho = \frac{3}{4} \Rightarrow \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$$

Mean service time = 10.5 minutes

$$\mu = \text{Mean service rate} = \frac{1}{\text{Mean service time}}$$

$$= \frac{1}{10.5} \text{ per minute}$$

$\frac{\lambda}{\mu} = \frac{3}{4}$

(9)

$$\lambda = \frac{3}{4}\mu$$

$$\lambda = \frac{3}{4}\left(\frac{1}{10.5}\right) = 0.0714 \text{ per minutes}$$

$\therefore$ Here, $\lambda = 0.0714$ - - - - - - (A)

$$\mu = \frac{1}{10.5} \quad \text{- - - - - - (B)}$$

$$\rho = \frac{3}{4} = 0.75 \quad \text{- - - - - - (C)}$$

$$\sigma = 8.8 \quad \text{- - - - - - (D)}$$

$$L_s = \left[\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu}$$

$$= \left[\frac{(0.0714)^2(8.8)^2 + (0.75)^2}{2(1-0.75)}\right] + \frac{3}{4} \quad \text{- - (1)}$$

$= 1.9145$

$= 1.9146 + 0.75$ $= 1.9145 + 0.75$

$= 2.6646$ $= 2.6645.$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$2.6645 - 0.75 = 1.9145$

$= 2.6646 - 0.75 = 1.9146 = 1.5797$ - - - (2)

$$W_s = \frac{1}{\lambda} L_s$$

$$= \left(\frac{1}{0.0714}\right)(2.6646) = 37.32 \quad \text{- - (3)}$$

$= 37.31$

$$W_q = \frac{1}{\lambda} L_q$$

$$= \left(\frac{1}{0.0714}\right)(1.9146)$$

$$= 26.815$$

I. (a) The average calling rate for the servi[ce] of the crane

$\lambda = 0.0714$ by (A)

(b) The average delay in getting service

$W_q = 26.815$ by (4)

II Given :

$\lambda = 0.0714$ min - - - - (A1)

$\mu = 1/8$ per min - - - - (B1)

$$\rho = \frac{\lambda}{\mu} = \frac{0.0714}{(1/8)} = 0.5712 \quad \text{- - (C1)}$$

$\sigma = 6$ min

$$L_s = \left[\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu}$$

$$= \frac{(0.0714)^2(6)^2 + (0.5712)^2}{2(1-0.5712)} + 0.5712$$

$$= 0.5944 + 0.5712$$

$$= 1.1656 \quad \text{- - - - (5)}$$

$L_q = L_s - \frac{\lambda}{\mu}$

$= 1.16556 - 0.5712 = 0.5944$ - - - (6)

$W_s = \frac{1}{\lambda} L_s$

$= \left(\frac{1}{0.0714}\right)(1.1656) = 16.325$ - - - (7)

$W_q = \frac{1}{\lambda} L_q$

$= \left(\frac{1}{0.0714}\right)(0.5944) = 8.325$ - - - (8)

(c) The reduction will occur on average, in the delay of getting served

$= 26.815 - 8.325$ by (4) & (8)

$= 18.5$ minutes.

Example 5.1.2

Consider a queueing system where arrivals are according to a poisson distribution with mean 5/hour. Find the expected waiting time in the system if the service time distribution is

(i) Uniform from t=5 min to t=15 min

(ii) Normal with mean 3 min. and variance 4 min²

Sol :-

This is an (M/G/1) queue model.

(i) Given :

Mean arrival rate (λ) $= 5$ per hour

$= 5 \times \frac{1}{60}$ per minutes

$= \frac{1}{12}$ per minutes

(i) The service time distribution is uniform from $t = 5$ min to $t = 15$ min

$\therefore a = 5, \ b = 15$

Mean service time $= \frac{a+b}{2} = \frac{5+15}{2} = 10$ min

Mean service rate (μ) $= \frac{1}{10}$ per minute

$\sigma^2 = Var(T) = \frac{(b-a)^2}{12} = \frac{(15-5)^2}{12} = 8.33$

$\rho = \frac{\lambda}{\mu} = \frac{\left(\frac{1}{12}\right)}{\left(\frac{1}{10}\right)} = 0.833$

$L_s = \left[\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu} = \frac{\left(\frac{1}{12}\right)^2(8.33) + (0.8}{2(1-0.833)}$

$= \frac{0.7517}{0.334} + 0.833$

$= 2.2506 + 0.833$

$= 3.084$

$W_s = \frac{1}{\lambda} L_s$

$= \frac{1}{\left(\frac{1}{12}\right)}(3.084) = (12)(3.084) = 37$ m

(ii) The service time distribution is Normal with mean 3 min. and Variance 4 min^2.

Given :

Mean arrival rate (λ) = $\frac{1}{12}$ per minutes

Mean service time = 3 min

Mean service rate (μ) = $\frac{1}{3}$ per min

$$\sigma^2 = 4$$

$$\rho = \frac{\lambda}{\mu} = \frac{(1/12)}{(1/3)} = 1/4$$

$$L_s = \left[\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu}$$

$$= \frac{(1/12)^2(4) + (1/4)^2}{2(1-1/4)} + \frac{1}{4}$$

$$= \frac{0.0903}{1.5} + 1/4$$

$$= 0.0602 + 1/4$$

$$= 0.310$$

$$W_s = 1/\lambda \, L_s$$

$$= 1/\left(\frac{1}{12}\right) (0.310)$$

$$= (12)(0.310)$$

$$= 3.72 \text{ min.}$$

TYPE 2: Problems based on M/D/1 model [In Type $\sigma^2 = 0$] (14)

Example 5.1.4

A car manufacturing plant uses one bi[g] crane for loading cars into a track. Car[s] arrive for loading by the crane according [to] a poisson distribution with a mean of 5 c[ars] per hour. Given that the service time for a[ll] cars is constant and equal to 6 minutes, det[ermine] L_s, L_q, W_s and W_q

The given problem is in $(M/G/1):(\infty/F$[IFO]) model

Given:

Mean arrival rate $[\lambda]$ = 5 per hour --- (A)

Mean service time = 6 min $= 6 \times \frac{1}{6} h$ =

Mean service rate $[\mu]$ $= \dfrac{1}{\text{Mean service time}}$

$= \dfrac{1}{(1/10)}$ per hour

= 10 per hour --- (B)

$\rho = \lambda/\mu$

$= 5/10$

$= 1/2$ --- (C)

The service time T is constant then var (T)

1 --- (D)

(18)

$$L_s = \left[\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu}$$

$$= \frac{0 + (1/2)^2}{2(1 - 1/2)} + 1/2$$

$$= \frac{(1/4)}{(1)} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} = 0.75 \quad \text{---(1)}$$

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} = 0.25 \quad \text{---(2)}$$

$$W_s = \frac{1}{\lambda} L_s = \left(\frac{1}{5}\right)(0.75) = 0.15 \text{ hr}$$

$$= (0.15)(60) \text{ min} = 9 \text{ minutes} \quad \text{---(3)}$$

$$W_q = \frac{1}{\lambda} L_q = \left(\frac{1}{5}\right)(0.25) = 0.05 \text{ hr}$$

$$= (0.05)(60) \text{ min}$$

$$= 3 \text{ minutes} \quad \text{----(4)}$$

Example 5.1.5

A one man barber shop takes exactly 25 minutes to complete one haircut. If customer arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer spends in the shop? Also find the average time a customer must wait for service.

Sol:

The average arrival time = 40 minutes

The average arrival rate [λ]

$$= \frac{1}{\text{The average arrival time}} = \frac{1}{40} \text{ per minute}$$

The average service time = 25 minutes

The average service rate [μ]

$$= \frac{1}{\text{The average service time}} = \frac{1}{25} \text{ per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{(1/40)}{(1/25)} = 5/8$$

If the service time T is constant, then σ^2

$$L_s = \left[\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu}$$

$$= \frac{0 + (5/8)^2}{2(1-5/8)} + 5/8 = \frac{(25/64)}{2(3/8)} + \frac{5}{8}$$

$$= \frac{(25/64)}{(6/8)} + 5/8 = \frac{25}{48} + \frac{5}{8} = \frac{55}{48}$$

$$L_q = L_s - \lambda/\mu = \frac{55}{48} - \frac{5}{8} = \frac{25}{48}$$

$$W_s = \frac{1}{\lambda} L_s = \frac{1}{(1/40)} \left(55/48\right) = (40)\left[\right.$$

$$= 45.8$$

$$W_q = \frac{1}{\lambda} L_q$$

$$= \frac{1}{(1/40)}\left(\frac{25}{48}\right) = (40)\left(\frac{25}{48}\right) = \frac{125}{6} = 20.8$$

Hence, a customer has to spend 45.8 minutes [in the] shop and has to wait for 20.8 minutes on the

TYPE 3: Problems based on Type 1 & Type 2

Example 5.17

An automatic car wash facility operates with only one bay. Cars arrive according to a poisson distribution with a mean of 4 cars/hr. and may wait in the facility's parking lot if the bay is busy. Find L_s, L_q, W_s, W_q if the service time

(a) is constant and equal to 10 minutes
(b) follows uniform distribution between 8 and 12 minutes.
(c) Follows normal distribution with mean 12 minute and S.D 3 minutes.
(d) Follows a discrete distribution with values 4, 8 and 15 minutes with corresponding probabilities 0.2, 0.6 and 0.2.

Sol

This is an M/G/1 queue model

Given: $\lambda = 4$ per hour

(a) Service time $= 10$ minutes $= \frac{1}{6}$ hours $= E[T]$

Since the service time is constant, variance $\sigma^2 = 0$, [M/D/1] model

$$\mu = \frac{1}{E(T)} = \frac{1}{(1/6)} = 6 \text{ per hour}$$

Since the service time is constant, variance $\sigma^2 = 0$, [M/D/1] model

$$\mu = \frac{1}{E(T)}$$

$$\rho = \frac{\lambda}{\mu} = \frac{4}{6} = \frac{2}{3}$$

by Pollachek - Khinchine formula

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = \frac{2}{3} + \frac{0 + (2/3)^2}{2(1 - 2/3)}$$

$$= 2/3 + \frac{4/9}{2(1/3)} = \frac{2}{3} + \left(\frac{4}{9}\right) \times \left(\frac{3}{2}\right)$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$L_q = L_s - \frac{\lambda}{\mu}, = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

$$W_s = \frac{L_s}{\lambda}, = \frac{(4/3)}{4} = \frac{1}{3} \text{ hr}$$

$$W_q = \frac{1}{\lambda} L_q, = \frac{1}{4}\left(\frac{2}{3}\right) = \frac{1}{6} \text{ hr}$$

(b) When service time follows uniform distribution between 8 and 12,

Here, $a = 8$, $b = 12$

$$E[T] = \frac{a+b}{2} = \frac{8+12}{2} = \frac{20}{2} = 10 \text{ min}$$

$$\mu = \frac{1}{E[T]} = \frac{1}{10} \text{ per minute},$$

$$\lambda = 4 \text{ per hour}$$

$$= \frac{1}{15} \text{ per minute}$$

$$\sigma^2 = V(T) = \frac{(b-a)^2}{12}$$

$$= \frac{(12-8)^2}{12} = \frac{4}{3}$$

(17)

$$\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/10} = \frac{1}{15} \times \frac{10}{1} = \frac{10}{15} = \frac{2}{3}$$

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

$$= \frac{2}{3} + \frac{(1/15)^2(4/3) + (2/3)^2}{2\left(1 - 2/3\right)} = \frac{2}{3} + \frac{(1/225)(4/3) + (4/9)}{2(1/3)}$$

$$= \frac{2}{3} + \frac{(4/675) + (4/9)}{(2/3)} = \frac{2}{3} + \left(\frac{304}{675}\right)\left(\frac{3}{2}\right)$$

$$= \frac{2}{3} + \frac{152}{225} = \frac{302}{225} = 1.34 \text{ car}$$

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{302}{225} - \frac{2}{5} = \frac{152}{225} = 0.676 \text{ car}$$

$$W_s = \frac{L_s}{\lambda} = \frac{(302/225)}{(1/15)} = \left(\frac{302}{225}\right)\frac{(15)}{1} = 20.13 \text{ min}$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.676}{0.067} = 10.09$$

i.e, a customer has to spend 20 minutes in the system and 10 minutes in the queue.

(c) When service time follows normal distribution with

$\mu = 1/12$ per minute

$\sigma = 3$ min, $\lambda = 1/15$ per minute

$$\rho = \frac{\lambda}{\mu} = \frac{(1/15)}{(1/12)} = \left(\frac{1}{15}\right)\left(\frac{12}{1}\right) = \frac{12}{15} = \frac{4}{5} \text{ min}$$

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

$$= \frac{4}{5} + \frac{(1/15)^2(3)^2 + (4/5)^2}{2(1-4/5)}$$

$$= \frac{4}{5} + \frac{(1/225)(9) + (16/25)}{2(1/5)}$$

$$= \frac{4}{5} + \frac{(17/25)}{(2/5)} = \frac{4}{5} + \left(\frac{17}{25}\right)\left(\frac{5}{2}\right)$$

$$= \frac{4}{5} + \frac{17}{10} = \frac{25}{10} = 2.5 \text{ cars}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= \frac{25}{10} - \frac{4}{5} = \frac{17}{10} = 1.7 \text{ cars}$$

$$W_s = \frac{1}{\lambda} L_s$$

$= \frac{15 \times 5}{2}$, $= \frac{75}{2} = 37.5$

$$= \frac{1}{(1/15)}\left(\frac{25}{10}\right) = 15\left(\frac{25}{10}\right) = \frac{375}{10} = 37.5 \text{ min}$$

$$W_q = \frac{1}{\lambda} L_q = \frac{1}{(1/15)}\left(\frac{17}{10}\right)$$

$$= 15\left(\frac{17}{10}\right) = \frac{255}{10} = 25.5 \text{ min}$$

(d)

$T = t$	4	8	15
$P(t)$	0.2	0.6	0.2

$$E(T) = \sum t\,p(t)$$

$$= (4)(0.2) + (8)(0.6) + (15)(0.2)$$

$$= 8.6 \qquad = 0.8 + 4.8 + 3.0$$

$E(T^2) = \sum t^2 p(t)$

$= (16)(0.2) + (64)(0.6) + (225)(0.2)$

$= 86.6$

$\sigma^2 = E[T^2] - [E[T]]^2$

$= 86.6 - (8.6)^2$

$= 12.64$

$\mu = \frac{1}{E[T]} = \frac{1}{8.6}$

$\rho = \frac{\lambda}{\mu} = \frac{(1/15)}{(1/8.6)} = 0.573$

$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$

$= 0.573 + \frac{(1/15)^2(12.64) + (0.573)^2}{2(1-0.573)}$

$= 1.023$ car

$L_q = L_s - \frac{\lambda}{\mu}$

$= (1.023) - 0.573 = 0.4502$

$W_s = L_s/\lambda = \frac{1.023}{(1/15)} = (1.023)(15)$

$= 15.345$ min

$W_q = L_q/\lambda$

$= \frac{0.4502}{(1/15)} = (0.4502)(15)$

$= 6.753$ min

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TYPE 4 : problems based on M/E_k/1 model

Example 5.1.8

A hospital clinic has a doctor examining every patient for a general checkup. On an average, the doctor takes 4 min in each phase of checkup. Although the distribution of time spent in each phase is exponential, if the patient goes through 4 phases (Er-log model) in checkup and the arrival of the patient to the doctor's service is PD at an average rate of 3/hr. Determine

(a) What is the average waiting time spent by a patient.

(b) Determine the expected number of patients in the system.

(c) What is the average time spent in the examination?

(d) What is the most probable time spent in the examination?

Sol

Given : $K = 4$

Service time in each phase $= 4$ min

Total service time $= 4 \times 4 = 16$ min

Mean service time $\mu = \frac{1}{16}$ per min

$= \frac{1}{16} \times 60$ per hour

$= 3.75$ per hour

$\lambda = 3$ per hour

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu-\lambda)}\left[\frac{k+1}{2k}\right]$$

$$= \frac{3}{3.75} + \frac{(3)^2}{(3.75)(3.75-3)}\left[\frac{4+1}{2(4)}\right]$$

$$= \frac{3}{3.75} + \frac{9}{2.8125}\left(\frac{5}{8}\right)$$

$$= \frac{3}{3.75} + \frac{45}{22.5} = 0.8 + 2 = 2.8 \text{ customers} \quad \text{--- (1)}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= 2.8 - \frac{3}{3.75} = 2.8 - 2 = 2 \quad \text{--- (2)}$$

$$W_s = \frac{1}{\lambda} L_s = \frac{1}{3}(2.8)$$

$$= 0.933 \text{ hr} \quad \text{--- (3)}$$

$$W_q = \frac{1}{\lambda} L_q = \frac{1}{3}(2)$$

$$= 0.667 \text{ hr} \quad \text{--- (4)}$$

$$\text{Mode} = \frac{k-1}{\mu k} = \frac{4-1}{(3.75)(4)} = \frac{3}{15} = 0.2 \text{ hr} \quad \text{--- (5)}$$

(a) What is the average waiting time spent by a patient?

$W_q = 0.667$ by (4)

(b) Determain the expected number of patients in the system.

$L_s = 2.8$ by (1)

(c) What is the average time spent in the examination?

Average time spent in examination

$= W_s - W_q$

$= 0.933 - 0.667 = 0.266 \times 60$ min by (3)

$= 15.96 \approx 16$ min

(d) What is the most probable time spent in the examination?

Mode $= 0.8$ hr by (5)

Example 5.1.10

In factory cafteria, the customers have to pass through 3 counters. The customers buy coupons at the 1st counter, select and collect snacks at the 2nd counter and collect tea at the 3rd counter. The server at each counter takes on an average 1.5 min although the distribution of service time is approximately exponential. If arrival of customers in approximately poisson, at an average rate of 6/hr, then calculate:

(a) The average time of the customers spend in the cafteria. W_s

(b) The average time of the customers of getting the service W_q

(c) The most probable time in getting the service. mode

Sol

Given

$K = 3$

Service time at one counter = 1.5 min

∴ Total service time = 4.5 min

$$\mu = \frac{1}{4.5} \times 60 = 13.33/hr$$

$$\lambda = 6/hr$$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu-\lambda)}\left(\frac{k+1}{2k}\right)$$

$$= \frac{6}{13.33} + \frac{(6)^2}{(13.33)(13.33-6)}\left[\frac{3+1}{2(3)}\right]$$

$$= \frac{6}{13.33} + \frac{36}{97.709}\left(\frac{4}{6}\right)$$

$$= 0.4561 + 0.2456$$

$$= 0.6957 \quad ---(1)$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= 0.6957 - 0.4501$$

$$= 0.2456 \quad ---(2)$$

$$* \; W_s = \frac{1}{\lambda} L_s = \frac{1}{6}(0.6957)$$

$$= 0.11595 \text{ hr} \quad --(3)$$

$$* \; \text{Mode} = \frac{k-1}{\mu k} = \frac{3-1}{(13.33)(3)} = \frac{2}{39.9}$$

$$= 0.05 \text{ hr} \quad ---(5)$$

(a) Average waiting time customer spend in cafeteria

$W_s = 0.1159$ hr by (3)

(b) Average time getting the service

$W = W_s - W_q = 0.115 - 0.0409$ by (3) & (4)

$= 0.0741$ hr $= 0.0741 \times 60$ min $= 4.5$ min

(c) Most probable time (5) Mode = 0.05 hr

Mode = 0.05 hr by ...

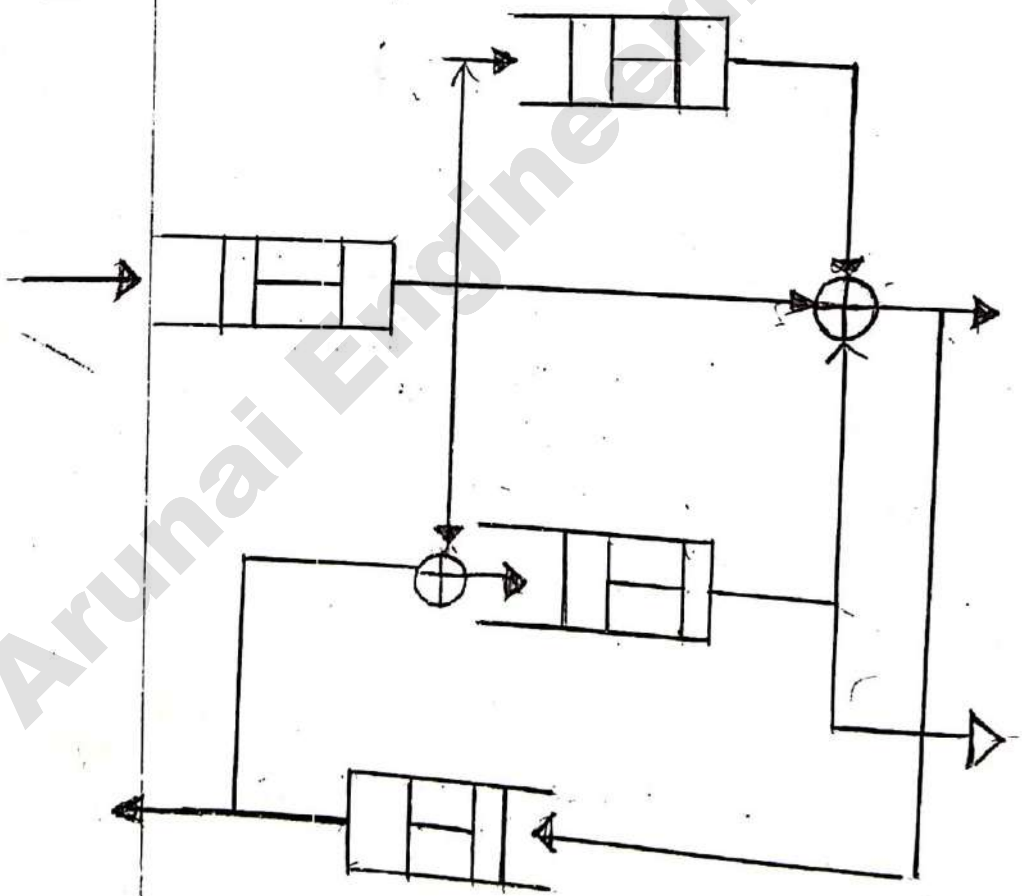
SERIES QUEUES

Classification of Queueing Networks

1. Open Networks
2. Closed Networks
3. Mixed Networks

1. Open Networks:-

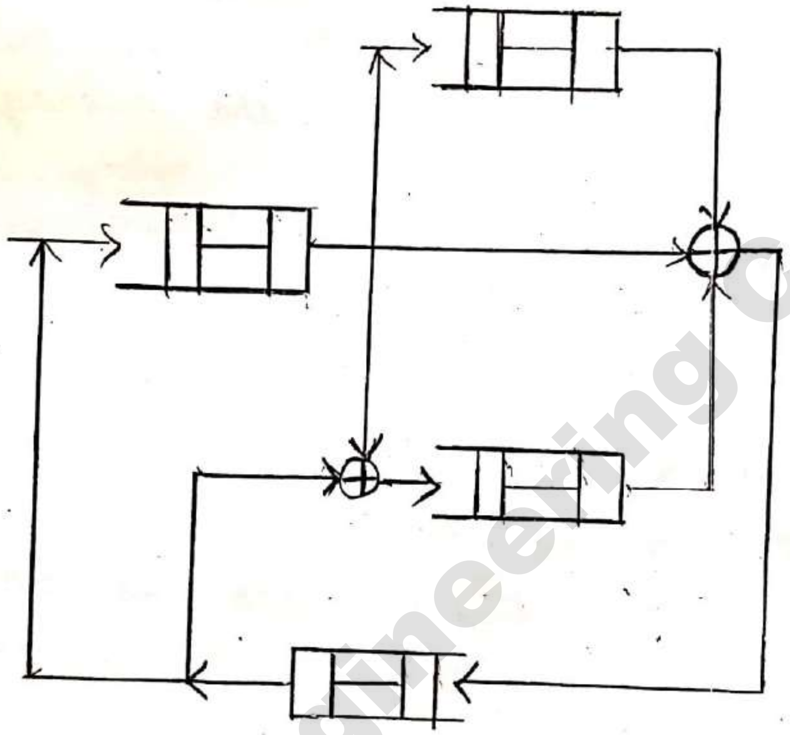
An open queuing network is characterised by one or more sources of job arrives and corresponding one or more sinks that absorb jobs departing from the network. If the network has multiple job classes then it must be open for each class of jobs



Open Queueing Network

2. Closed Network :-

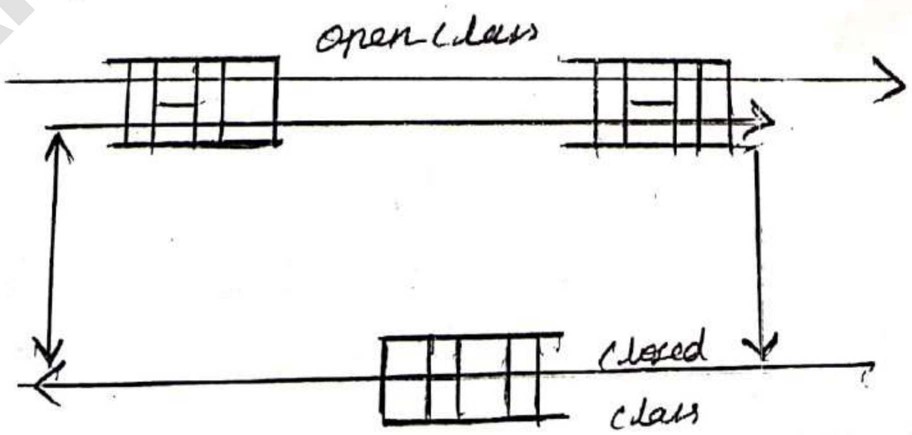
In a closed queuing network, jobs neither enter nor depart from the network. If the network has multiple job classes then it must be closed for each class of jobs



Closed Queueing Network

3. Mixed Network :-

Network has multiple job classes and is open with respect to some classes but closed with repeat to the others.



Mixed Network

(2)

1. problems under single server queueing System

Example 5.3.1

In a big factory, there are a large number of operating machines and two sequential repair shops, which do the service of the damaged machines exponentially with respective rates of 1/hour and 2/hour. If the cumulative failure rate of all the machines in the factory is 0.5/hour, find (i) the probability that both repair shops are idle, (ii) the average number of machines in the service section of the factory and (iii) The average repair time of a machine.

Sol

Given : $\lambda = 0.5/\text{hour}$

$= \frac{1}{2}$ per hour

$\mu_1 = 1$ per hour

$\mu_2 = 2$ per hour

The situation in this problem is comparable with 2-stage Tandem queue with single server at each state.

(i) P(both the service stations are idle)

$$= P(0,0)$$

$$= \left(\frac{\lambda}{\mu_1}\right)^0 \cdot \left(1 - \frac{\lambda}{\mu_1}\right) \cdot \left(\frac{\lambda}{\mu_2}\right)^0 \cdot \left(1 - \frac{\lambda}{\mu_2}\right)$$

$$= \left(\frac{1/2}{1}\right)^0 \left(1 - \frac{1/2}{1}\right) \left(\frac{1/2}{2}\right)^0 \left[1 - \frac{1/2}{2}\right]$$

$$= (1/2)(1 - 1/4) = (1/2)(3/4) = (3/8)$$

(ii) The average number of machines in service

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda}$$

$$= \frac{1/2}{1-1/2} + \frac{1/2}{2-1/2} = \frac{1/2}{1/2} + \frac{1/2}{3/2} = 1 + \frac{1}{3} = \frac{4}{3}$$

(iii) The average repaire time

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{1-1/2} + \frac{1}{2-1/2} = \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{3}{2}\right)}$$

$$= 2 + \frac{2}{3} = \frac{8}{3}$$

Example 5.3.2

A repair facility shared a large number of machine has 2 sequential stations with respective service rates of 2 per hour and 3 per hour. The cumulative failure rate of all the machines is 1 per hour. Assuming that the system behaviour may be approximated by the 2-stage tandem queue, find (i) the average repair time including the waiting time. (ii) the probability that both the service stations are idle and (iii) the bottleneck of the repair facility.

(Or)

A TVS company in Madurai containing repair facility shared by a large number of machines has 2 sequential stations with respective service rates of 2 per hour and 3 per hour. The cumulative failure rate of all the machines is 1 per hour. Assuming that the system behaviour may be approximated by the 2-stage tandem queue, find (i) the average repair time including the waiting time (ii) the probability that both the service stations are idle and (iii) the bottleneck of repair facilities.

Sol

Given : $\lambda = 1$, $\mu_1 = 2$ and $\mu_2 = 3$

The situation in this problem is comparable with 2- stage Tandem queue with single server at each state.

(i) The average number of machines in service

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda} = \frac{1}{2-1} + \frac{1}{3-1} = 1 + \frac{1}{2}$$

$$= 3/2$$

The average repair time including the waiting time $= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{2-1} + \frac{1}{3-1} = 1 + \frac{1}{2} = \frac{3}{2}$

(ii) P(both the service stations are idle)

$$= P(0,0) = \left(\frac{\lambda}{\mu_1}\right)^0 \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^0 \left(1 - \frac{\lambda}{\mu_2}\right)$$

$$= (1)\left(1 - \frac{1}{2}\right)(1)\left(1 - \frac{1}{3}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 1/3$$

(iii) Here $\frac{\lambda}{\mu_1} = \frac{1}{2}$, $\frac{\lambda}{\mu_2} = \frac{1}{3}$

$$\frac{\lambda}{\mu_1} > \frac{\lambda}{\mu_2}$$

$\therefore$ the service station 1 is the bottleneck of the repair facilities.

Example 5.3.3

In a clinic, there are two sections - one section for assessing first time and the other for final assessment and prescription. patients arrive a clinic in a poisson fashion at the rate of 3 per hour. The assistant in the first section takes nearly 15 minutes per patient for approximate assessment and the doctor in the second section takes nearly 6 minutes per patient for final prescription. If the service time in the two sections are approximately exponential, find the probability that there are 3 patients in the first section and 2 patients in the second section. Find also the average number of patient in the clinic and the average waiting time of a patient in the clinic. Assume that enough space is available for the patients to wait in front of both sections.

Sol

Given:-

$\lambda = 3$ per hour, $\mu_1 = 4$ per hour, $\mu_2 = 10$ per hour

The situation in this problem is comparable with 2-stage Tandem queue with single server at each other.

(i) P(3 customers in the first section and 2 customers in the second) i.e., P(2,3)

We know that,

$$p(m,n) = \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^n\left(1-\frac{\lambda}{\mu_2}\right)$$

$$p(3,2) = \left(\frac{3}{4}\right)^3\left(1-\frac{3}{4}\right)\left(\frac{3}{10}\right)^2\left(1-\frac{3}{10}\right)$$

$$= \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)^2\left(\frac{7}{10}\right)$$

$$= \left(\frac{27}{64}\right)\left(\frac{1}{4}\right)\left(\frac{9}{100}\right)\left(\frac{7}{10}\right) = \frac{1701}{256000}$$

(ii) The average number of patients in the clinic

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda} = \frac{3}{4-3} + \frac{3}{10-3}$$

$$= \frac{3}{1} + \frac{3}{7} = \frac{24}{7}$$

(iii) The average waiting time of a patient in the clinic.

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{4-3} + \frac{1}{10-3}$$

$$= \frac{1}{1} + \frac{1}{7} = \frac{8}{7} \text{ hours.}$$

II Problems under multiple server queueing system

Example 5.3.4.

In the Airport reservation section of a city junction, there is enough space for the customers to assemble, form a queue and fill up the resorvation forms. There are 5 reservation counters in front of which also there is enough space for the customers to wait. Customers arrive at the reservation section at the rate of 60 / hour according to poisson process, take 1 minute each on the average to fill up the forms and then move to the reservation counter section. Each reservation clerk takes 5 minutes on the average to complete the business of a customer in an exponential manner.

i) Find the probability that a customer has to wait to get the service in the reservation counter section.

ii) Find the total waiting time for a customer in the entire reservation section. Assume that only those who have the filled up reservation forms will be allowed into the counter section.

Sol

Filling up the reservation forms only the customer can enter the reservation counter section, it is a sequential queuing model.

I. The reservation form section is a self-service station

∴ It is an $(M/M/1):(\infty/FCFS)$ queuing model.

II There are 5 reservation counters in the second station.

∴ It is an $(M/M/s):(\infty/FCFS)$ queuing model

I. Given : $\lambda = 50$ per hour

$\mu = 60$ per hour

$$L_{s1} = \frac{\lambda}{\mu-\lambda} = \frac{50}{60-50} = \frac{50}{10} = 5 \text{ per hour}$$

$$W_{s1} = \frac{1}{\lambda} L_{s1} = \left(\frac{1}{50}\right)(5) = \frac{1}{10} \text{ per hour} \quad \text{- - - - - - (A)}$$

II Given : $\lambda = 50$ per hour $\quad$ - - - - - (B)

$\mu = 12$ per hour

$s = 5$

$$\rho = \frac{\lambda}{s\mu} = \frac{50}{(5)(12)} = \frac{50}{60} = \frac{5}{6} = 0.83 \quad \text{- - - (C)}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{(\lambda/\mu)^s}{s!(1-\rho)}\right]^{-1}$$

Take : $\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n = \sum_{n=0}^{4} \frac{1}{n!} (4.17)^n$

$= 1 + \frac{4.17}{1!} + \frac{(4.17)^2}{2!} + \frac{(4.17)^3}{3!} + \frac{(4.17)^4}{4!}$

$= 1 + 4.17 + 8.69 + 12.09 + 12.60 = 38.55$

Take : $\frac{(\lambda/\mu)^s}{s!(1-\rho)} = \frac{(4.17)^5}{5!(1-0.83)} = \frac{1260.9}{20.4} = 61.81$

$\therefore P_0 = [38.55 + 61.81]^{-1} = (100.36)^{-1} = 0.00$

$L_{s2} = \frac{1}{ss!} \frac{(\lambda/\mu)^{s+1}}{(1-\rho)^2} P_0 + \frac{\lambda}{\mu}$

$= \frac{1}{(5)(5!)} \frac{(4.17)^6}{(1-0.83)^2} (0.009) + 4.17$

$= \frac{47.32}{17.34} + 4.17 = 6.899$

$W_{s2} = 1/\lambda \; L_{s2} = 1/50 \; (6.899) = 0.14 \; hr$

(i) P(a customer has to wait in the counter section)

$= \frac{1}{s!} \frac{(\lambda/\mu)^s}{(1-\rho)} P_0$

$= \frac{1}{5!} \frac{(4.17)^5}{(1-0.83)} (0.009) = 0.55$

(ii) The total waiting time for a customer in the entire reservation $= W_{s1} + W_{s2} = \frac{1}{10} + 0.14 = 0.1 + 0.14$

$= 0.24 \; h = (0.24)(60) \; min = 14.4 \; min$

Note :

(iii) Total number of customers in the entire reservation room $= L_{s1} + L_{s2}$

$= 5 + 6.899 = 11.899 \; h.$

Example 5.4.1

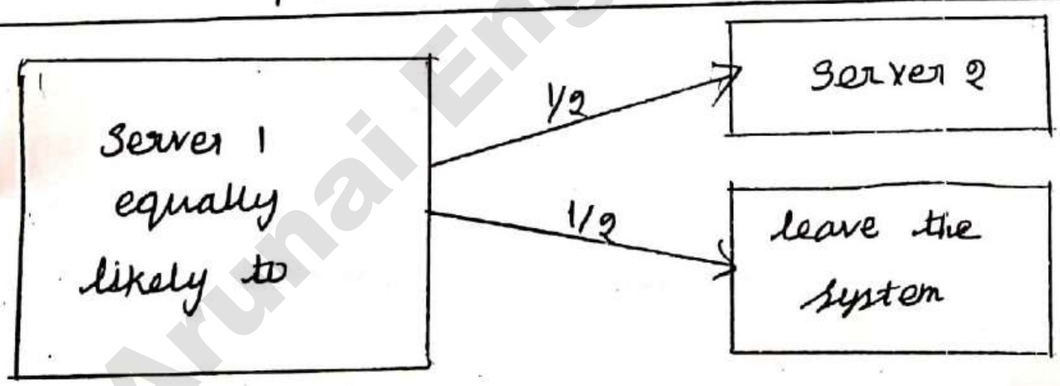
Consider a system of two servers where customers from outside the system arrive at server 1 at a poisson rate 4 and at server 2 at a poisson rate 5. The service rates 1 and 2 are respectively 8 and 10. A customer upon completion of service at server 1 is equally likely to go to server 2 or to leave the system (i.e., $P_{11}=0$, $P_{12}=1/2$); where as a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise (i.e.,) $P_{21}=1/4$, $P_{22}=0$). Determain the limiting probabilities, L_s and W_s

Sol

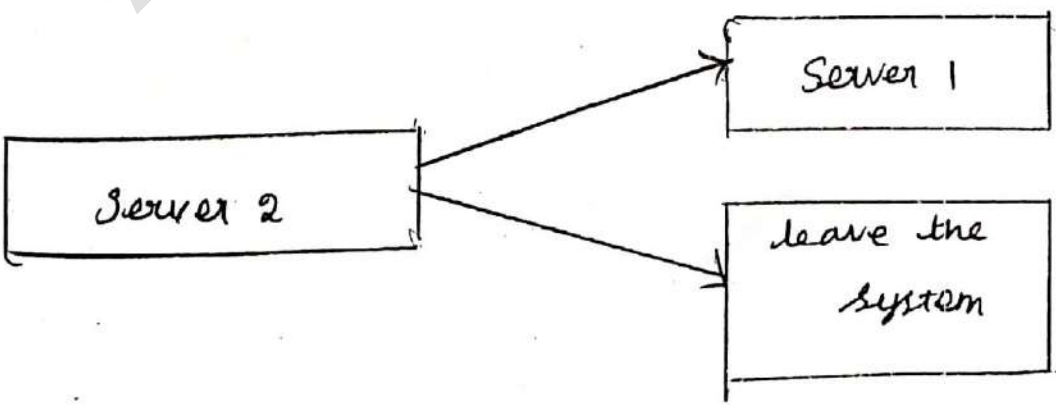
The given system is a Jackson's open queue system. Let λ_1, λ_2 be the total arrival rates at S_1 and S_2 respectively.

Given: $r_1 = 4$, $r_2 = 5$

$\mu_1 = 8$, $\mu_2 = 10$



$\Rightarrow \begin{cases} P_{11} = 0 \\ P_{12} = 1/2 \end{cases}$



$\Rightarrow \begin{cases} P_{21} = \frac{25}{100} = 1/4 \\ P_{22} = 0 \end{cases}$

(35)

The Jackson's flow balance equations are

$$\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i P_{ij}, \; j = 1, 2 \quad \text{- - - - (1)}$$

For $j=1$, we get

$$\lambda_1 = r_1 + \sum_{i=1}^{2} \lambda_1 P_{i1}$$

i.e., $\lambda_1 = r_1 + \lambda_1 P_{11} + \lambda_2 P_{21}$

$$= 4 + \lambda_1 (0) + \lambda_2 \left(\frac{1}{4}\right)$$

$$\lambda_1 = 4 + \lambda_2/4$$

$$\lambda_1 = 1/4 \lambda_2 = 4 \quad \text{- - - - (2)}$$

For $j=2$, we get

$$\lambda_2 = r_2 + \sum_{i=1}^{2} \lambda_1 P_{i2}$$

i.e., $\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22}$

$$= 5 + \lambda_1 (1/2) + \lambda_2 (0)$$

$$\lambda_2 = 5 + 1/2 (\lambda_1)$$

$$-\frac{1}{2}\lambda_1 + \lambda_2 = 5 \quad \text{- - - - - - . (3)}$$

Solving (2) & (3) by using your calculator fx 991MS, we get

$$\lambda_1 = 6, \; \lambda_2 = 8.$$

Given : $\mu_1 = 8$, $\mu_2 = 10$, $\frac{\lambda_1}{\mu_1} = \frac{6}{8} = \frac{3}{4}$,

$$\frac{\lambda_2}{\mu_2} = \frac{8}{10} = \frac{4}{5}$$

Hence, P[n_1 at server 1, n_2 at server 2]

$$p[n_1, n_2] = \left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left[1-\frac{\lambda_1}{\mu_1}\right]\left[\frac{\lambda_2}{\mu_2}\right]^{n_2}\left[1-\frac{\lambda_2}{\mu_2}\right]$$

$$p(1,2) = \left(\frac{3}{4}\right)^1\left(1-\frac{3}{4}\right)\left(\frac{4}{5}\right)^2\left(1-4/5\right)$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{16}{25}\right)\left(\frac{1}{5}\right) = 3/125$$

$$L_s = \frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\lambda_2}{\mu_2-\lambda_2}$$

$$= \frac{6}{8-6} + \frac{8}{10-8} = \frac{6}{2} + \frac{8}{2} = 3+4 = 7$$

$$W_s = 1/\lambda \, L_s = 1/9\,(7) = 7/9$$

$[\because \lambda = 4+5]$

Example 5.4.3

Consider two servers. An average of 8 customer per hour arrive from outside at server 1 and an average of 17 customers per hour arrive from outside at server 2. Interarrival times are exponential. Server 1 can serve at an exponential rate of 20 customers per hour and server 2 can serve at an exponential rate of 30 customer per hour. After completing service at server 1, half of the customers leave the system, and half to go to server 2. After completing service at server 2, 3/4 of the customers complete service, and $\frac{1}{4}$ return to server 1.

Inaction of the time is server 1

(ii) Find the expected number of customers at each server.

(iii) Find the average time a customer spends in the system.

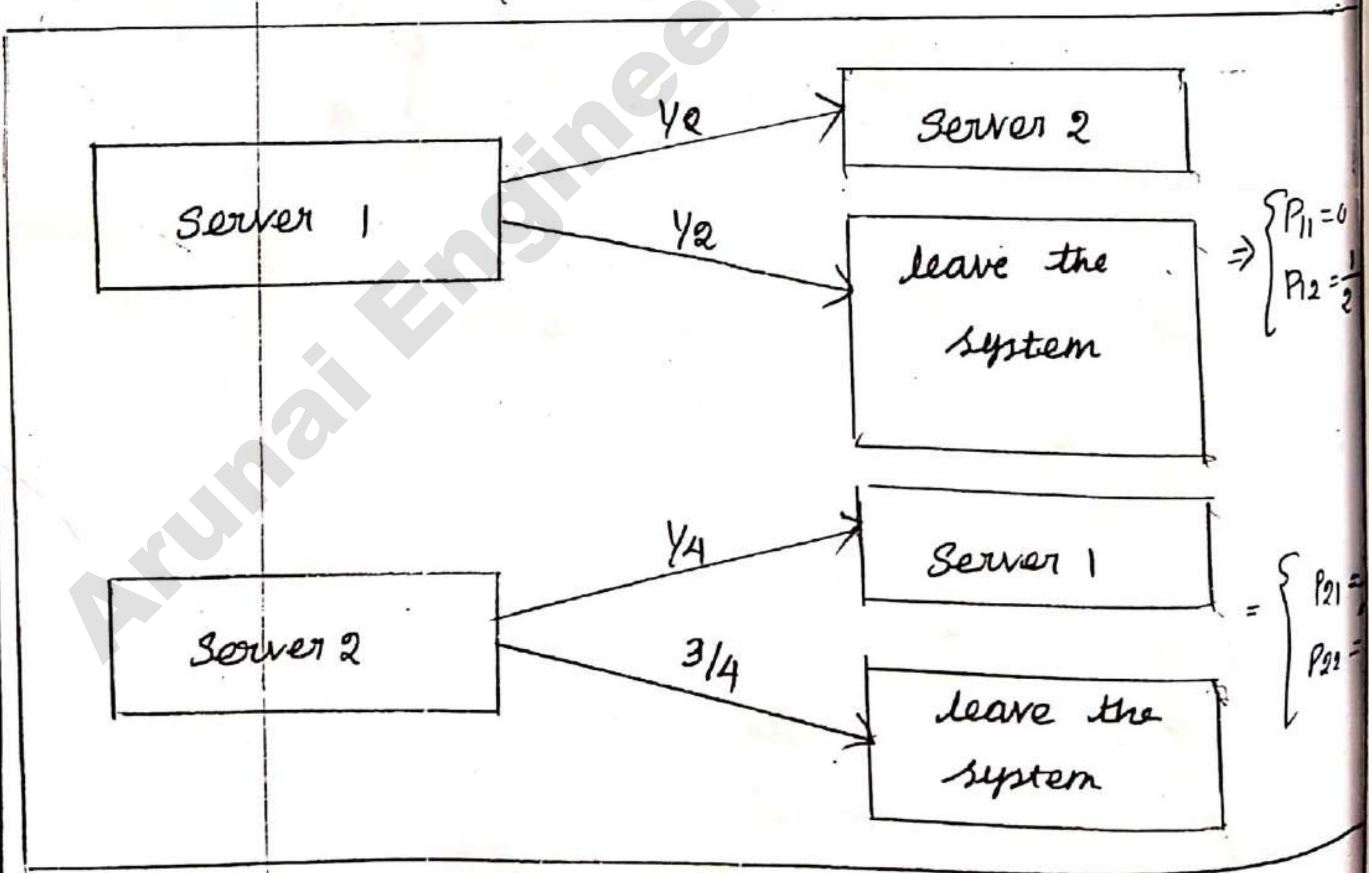
(iv) How would the answers to parts (i) – (iii) change if server 2 would server only an average of 20 customers per hour?

Sol :-

The given system is a Jackson's open queue system.

Let λ_1, λ_2 be the total arrival rates at s_1 and s_2 respectively.

Given: $r_1 = 8, r_2 = 17, \mu_1 = 20, \mu_2 = 30$



The Jackson's flow balance equations are

$$\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i P_{ij}, \; j = 1, 2 \quad \text{------} (1)$$

For, $j=1$ we get

$$\lambda_1 = r_1 + \sum_{i=1}^{2} \lambda_j P_{i1}$$

$$\lambda_1 = r_1 + \lambda_1 P_{11} + \lambda_2 P_{21} = 8 + \lambda_1 (0) + \lambda_2 \left(\frac{1}{4}\right)$$

$$\lambda_1 = 8 + \frac{1}{4}\lambda_2 \Rightarrow \lambda_1 - \frac{1}{4}\lambda_2 = 8$$

$$\lambda_1 - \frac{1}{4}\lambda_2 = 8 \quad \text{------} (2)$$

For $j = 2$ we get

$$\lambda_2 = r_2 + \sum_{i=1}^{2} \lambda_i P_{i2}$$

i.e., $\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22}$

$$= 17 + \lambda_1 (1/2) + \lambda_2 (0)$$

$$\lambda_2 = 17 + 1/2 \lambda_1$$

$$-1/2 \lambda_1 + \lambda_2 = 17 \quad \text{------} (3)$$

Solving (2) & (3) by using your calculator fx 991MS, we get

$$\lambda_1 = 14, \quad \lambda_2 = 24$$

(i) Server 1 may be treated as an $(M/M/1/GP/\infty/\infty)$ system with $\lambda = 14$, $\mu = 20$

$$P_0 = 1 - \rho = 1 - \left(\frac{\lambda}{\mu}\right) = 1 - \left(\frac{14}{20}\right) = \frac{6}{20} = \frac{3}{10} = 0.3$$

Thus server 1 is idle 30% of the time.

(ii) $L_s = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2}$

$= \frac{14}{20-14} + \frac{24}{30-24} = \frac{14}{6} + \frac{24}{6} = \frac{7}{3} + 4 = \frac{19}{3}$

(iii) $W_s = \frac{1}{\lambda} L_s = \frac{1}{25}\left(\frac{19}{3}\right) = \frac{19}{75}$ $\left[\because \lambda = r_1 + r_2 = 8 + 17 = 25\right]$

$\frac{1}{25}(-3.0666) =$

(iv) $s_2 \mu_2 = 20 < \lambda_2$, so no steady state exists.

Example 5.4.5

In a network of 3 service stations 1, 2, 3 customers arrive 1, 2, 3 from outside, in accordance with Poisson process having rates 5, 10 respectively. The service times at the 3 stations are exponential with respective rates 10, 50, 100. A customer completing service at station 1 is equally like to (1) go to station 2, (2) go to station 3 and (3) leave the system.

A customer departing from service at station 2 always goes to station 3. A departure from service at station 3 is equally like to go to station 2 or leave the system.

(A) What is the average number of customers in the system consisting of all the three stations?

(B) What is the average time a customer spends in the system?

Sol:-

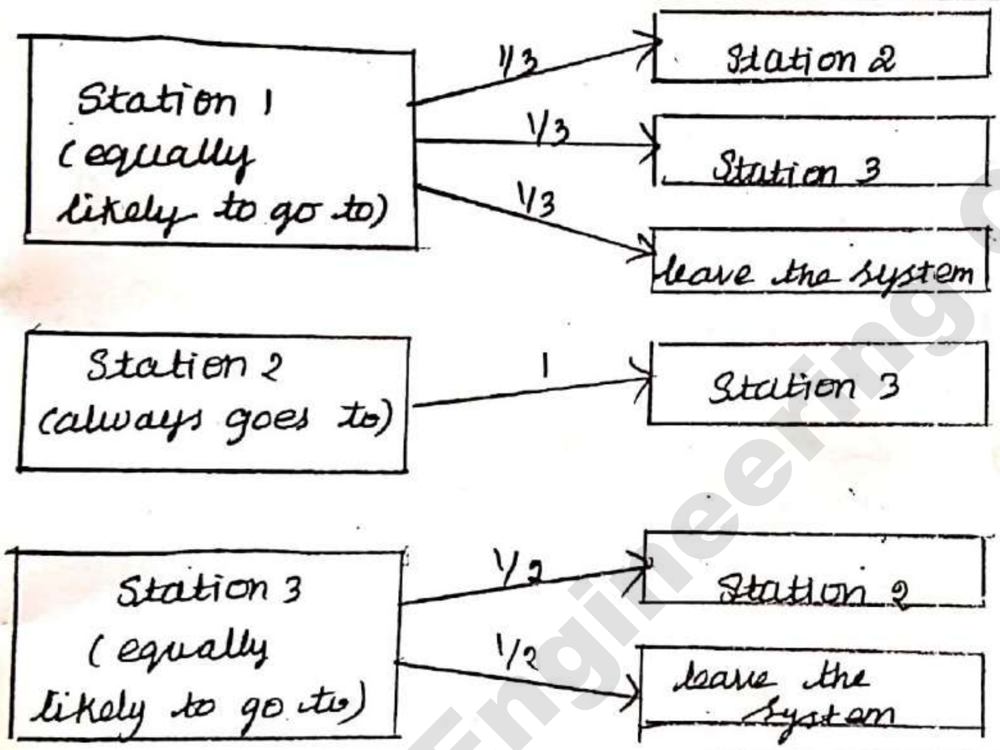
The given system is a Jackson's open queue system.

Let $\lambda_1, \lambda_2, \lambda_3$ be the total arrival rates at S_1, S_2 and S_3 respectively.

Given :

$r_1 = 5, \ r_2 = 10, \ r_3 = 15$

$\mu_1 = 10, \ \mu_2 = 50, \ \mu_3 = 100$



$\Rightarrow \begin{cases} P_{11} = 0 \\ P_{12} = 1/2 \\ P_{13} = 1/3 \end{cases}$

$\Rightarrow \begin{cases} P_{21} = 0 \\ P_{22} = 0 \\ P_{23} = 1 \end{cases}$

$\Rightarrow \begin{cases} P_{31} = 0 \\ P_{32} = 1/2 \\ P_{33} = 0 \end{cases}$

The Jackson's flow balance equations are

$$\lambda_j = r_j + \sum_{i=1}^{3} \lambda_i P_{ij}, \ j = 1, 2, 3 \quad \text{- - - - (1)}$$

For $j = 1$, we get

$$\lambda_1 = r_1 + \sum_{i=1}^{3} \lambda_i, P_{i1}$$

i.e., $\lambda_1 = r_1 + \lambda_1 P_{11} + \lambda_2 P_{21} + \lambda_3 P_{31}$

$= 5 + \lambda_1 (0) + \lambda_2 (0) + \lambda_3 (0)$ - - - - (2)

$\lambda_1 = 5$

For $j=2$, we get

$$\lambda_2 = r_2 + \sum_{i=1}^{3} \lambda_i P_{i2}$$

i.e., $\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22} + \lambda_3 P_{32}$

$$= 10 + \lambda_1 (1/3) + \lambda_2 (0) + \lambda_3 (1/2)$$

$$\lambda_2 = 10 + \lambda_1/3 + \lambda_3/2$$

$$\lambda_2 = 10 + 5/3 + \lambda_3/3 \qquad [\because \lambda_1 = 5]$$

$$\lambda_2 = \frac{35}{3} + \frac{\lambda_3}{2}$$

$$\lambda_2 - \frac{1}{2}\lambda_3 = \frac{35}{3} \qquad \text{-------- (3)}$$

For $j=3$, we get

$$\lambda_3 = r_3 + \sum_{i=1}^{3} \lambda_i P_{i3}$$

$$\lambda_3 = r_3 + \lambda_1 P_{13} + \lambda_2 P_{23} + \lambda_3 P_{33}$$

$$= 15 + \lambda_1 (1/3) + \lambda_2 (1) + \lambda_3 (0)$$

$$\lambda_3 = 15 + \lambda_1/3 + \lambda_2$$

$$\lambda_3 = 15 + 5/3 + \lambda_2 \qquad [\because \lambda_1 = 5]$$

$$\lambda_3 = 50/3 + \lambda_2$$

$$\lambda_2 - \lambda_3 = -50/3 \qquad \text{-------- (4)}$$

Solving (3) & (4) by using your calculator fx 991MS, we get

$$\lambda_2 = 40, \quad \lambda_3 = 170/3 = 56.67$$

(A) $L_s = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} + \frac{\lambda_3}{\mu_3 - \lambda_3}$

$= \frac{5}{10-5} + \frac{40}{50-40} + \frac{56.67}{100-56.67} = 6.31$

(B) $W_s = \frac{L_s}{\lambda} = \frac{6.31}{30} = 0.21$ $[\because \lambda = r_1 + r_2 + r_3 = 5 + 10 + 15 = 30]$

Example 5.4.7

Jackson network with three facilities that have the parameters given below $P_{11} = 0$, $P_{12} = 0.6$, $P_{13} = 0.3$, $P_{21} = 0.1$, $P_{22} = 0$, $P_{23} = 0.3$, $P_{31} = 0.4$, $P_{32} = 0.4$, $P_{33} = 0$, $\mu_1 = 10$, $\mu_2 = 10$, $\mu_3 = 10$, $C_1 = 1$, $C_2 = 2$, $C_3 = 1$, $r_1 = 1$, $r_2 = 4$, $r_3 = 3$

Find (i) The total arrival rate at each facility

(ii) $P(n_1, n_2, n_3)$

(iii) Expected number of customers in the entire system.

(iv) Expected time a customer spends in the system.

(Or)

For a open queuing network with three nodes 1, 2 and 3, let customer arrive from outside the system to node j according to a Poisson input process with parameters r_j and let P_{ij} denote the proportion of customers departing from facility i to facility j.

(A3)

Given $(r_1, r_2, r_3) = (1, 4, 3)$ and ~~departing~~ from ~~facilities~~.

$$P_{ij} = \begin{pmatrix} 0 & 0.6 & 0.3 \\ 0.1 & 0 & 0.3 \\ 0.4 & 0.4 & 0 \end{pmatrix}$$

determine the average arrival rate λ_j to the node j for $j = 1, 2, 3$.

Sol

Let $\lambda_1, \lambda_2, \lambda_3$ be the total arrival rate at facility 1, 2, 3 respectively.

Given : $r_1 = 1$, $r_2 = 4$, $r_3 = 3$

$\mu_1 = 10$, $\mu_2 = 10$, $\mu_3 = 10$

$P_{11} = 0$; $P_{12} = 0.6$, $P_{13} = 0.3$

$P_{21} = 0.1$, $P_{22} = 0$, $P_{23} = 0.3$

$P_{31} = 0.4$, $P_{31} = 0.4$, $P_{33} = 0$

Jackson's flow balance equation are

$$\lambda_j = r_j + \sum_{i=1}^{3} \lambda_i P_{ij}, \; j = 1, 2, 3 \quad \text{---} (1)$$

For $j = 1$, we get

$$\lambda_1 = r_1 + \sum_{i=1}^{3} \lambda_i P_{i1}$$

i.e., $\lambda_1 = r_1 + \lambda_1 P_{11} + \lambda_2 P_{21} + \lambda_3 P_{31}$

$= 1 + \lambda_1 (0) + \lambda_2 (0.1) + \lambda_3 (0.4)$

$\lambda_1 = 1 + (0.1)\lambda_2 + (0.4)\lambda_3$

$$\lambda_1 - (0.1)\lambda_2 - (0.4)\lambda_3 = 1 \quad \text{-----} (2)$$

For $j = 2$, we get

$$\lambda_2 = r_2 + \sum_{i=1}^{3} \lambda_i p_{i2}$$

$$\lambda_2 = r_2 + \lambda_1 p_{12} + \lambda_2 p_{22} + \lambda_3 p_{32}$$

$$= 4 + \lambda_1 (0.6) + \lambda_2 (0) + \lambda_3 (0.4)$$

$$\lambda_2 = 4 + (0.6)\lambda_1 + (0.4)\lambda_3 .$$

$$-(0.6)\lambda_1 + \lambda_2 - (0.4)\lambda_3 = 4 \quad \text{-------} \quad (3)$$

For $j = 3$, we get

$$\lambda_3 = r_3 + \sum_{i=1}^{3} \lambda_i p_{i3}$$

$$\lambda_3 = r_3 + \lambda_1 p_{13} + \lambda_2 p_{23} + \lambda_3 p_{33}$$

$$= 3 + \lambda_1 (0.3) + \lambda_2 (0.3) + \lambda_3 (0)$$

$$\lambda_3 = 3 + (0.3)\lambda_1 + (0.3)\lambda_2 .$$

$$(-0.3)\lambda_1 - (0.3)\lambda_2 + \lambda_3 = 3 \quad \text{-------} \quad (4)$$

Solving (2), (3) & (4) by using your calculator fx 991MS, we get $\lambda_1 = 5$, $\lambda_2 = 10$, $\lambda_3 = 7.5$

(ii) Facility 1 is an (M|M|1) model

Facility 2 is an (M|M|1) model

Facility 3 is an (M|M|1) model

Facility 1 : $\lambda_1 = 5$, $\mu_1 = 10$, $c_1 = 1$

$$P_{n_1} = \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(1 - \frac{\lambda_1}{\mu_1}\right)$$

$$= \left(\frac{5}{10}\right)^{n_1} \left(1 - \frac{5}{10}\right) = \left(\frac{1}{2}\right)^{n_1} \left(\frac{1}{2}\right)$$

$$L_{s1} = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{5}{10-5} = \frac{5}{5} = 1$$

Facility 2 : $\lambda_2 = 10$, $\mu_2 = 10$, $c_2 = 2$

$$P_{n2} = \begin{cases} \frac{1}{n_2!} \left(\frac{\lambda_2}{\mu_2}\right)^{n_1} P_0, & \text{if } n_2 < 2 \\ \frac{1}{c_2! \, c_2^{n_2 - c_2}} \left(\frac{\lambda_2}{\mu_2}\right)^{n_2} P_0, & \text{if } n_2 \geq 2 \end{cases}$$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!} \left(\frac{\lambda_2}{\mu_2}\right)^n + \frac{\frac{1}{2!} (\lambda_2/\mu_2)^2}{\left(1 - \frac{\lambda_2}{2\mu_2}\right)} \right]^{-1}$$

$$= \left[1 + \frac{1}{1!} \left(\frac{10}{10}\right)^1 + \frac{\frac{1}{2!} (10/10)^2}{(1/2)} \right]^{-1}$$

$$= [1+1+1]^{-1} = 1/3$$

$$P_1 = 1/1! \, (\lambda_2/\mu_2)^1 P_0 = 1/3$$

$$\therefore P_{n2} = \begin{cases} 1/3 & \text{if } n_2 = 0 \\ 1/3 & \text{if } n_2 = 1 \\ (1/2)(1/2)^{n_2 - 2}(1/3) & \text{if } n_2 \geq 2 \end{cases}$$

$$L_{s2} = \frac{(\lambda_2/\mu_2)^{c_2+1}}{c_2 (c_2!) \left(1 - \frac{\lambda_2}{c_2 \mu_2}\right)^2} P_0 + \frac{\lambda_2}{\mu_2}$$

$$= \frac{1}{(2)(2)(1/4)} (1/3) + 1 = \frac{1}{3} + 1 = \frac{4}{3}$$

Facility 3 : $\lambda_3 = 7.5$, $\mu_3 = 10$, $c_3 = 1$

$$P_{n3} = \left(\frac{\lambda_3}{\mu_3}\right)^{n_3} \left(1 - \frac{\lambda_3}{\mu_3}\right) = \left(\frac{7.5}{10}\right)^{n_3} \left(1 - \frac{7.5}{10}\right)$$

$= (0.75)^{n_3} (2.5/10)$

$= (0.75)^{n_3} (0.25)$

$$L_{s_3} = \frac{\lambda_3}{\mu_3 - \lambda_3} = \frac{7.5}{10-7.5} = \frac{7.5}{2.5} = 3$$

(iii) $L_s = L_{s_1} + L_{s_2} + L_{s_3} = 1 + \frac{4}{3} + 3$

$= 4 + \frac{4}{3} = \frac{16}{3}$

(iv) $W_s = \frac{L_s}{\lambda} = \frac{16/3}{8} = \frac{2}{3}$ $[\because \lambda = r_1 + r_2 + r_3 = 1 + 4 + 3 = 8]$

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2 marks :-

1) Write down Pollaczek-Khintchine formula and explain the notations.

If T is the random service time, the average number of customers in the system

$$L_s = E_n = \lambda E(T) + \frac{\lambda^2 [E^2(T) + V(T)]}{2[1 - \lambda E(T)]}$$

where $E(T)$ is mean of T

$V(T)$ is variance of T

2) Distinguish between open and closed network

Open network	Closed network
1. Arrivals from outside to the node $i (r_i)$ is allowed	New customers never enter in to the system
2. Once a customer gets the service completed at node i, he joins the queue at node j with probability P_{ij} or leaves the system with probability P_{i0}	Existing customers never depart from the system (i.e) $P_{i0} = 0$ and $r_i = 0$ for all i (OR) No customer may leave the system.

3) Explain (series queue) tandem queue model:-

Sol : A series queue model or a tandem queue model is satisfies the following characteristics

(i) Customers may arrive from outside the system at any node and may leave the system from any node.

(ii) Customers may enter the system at some node, traverse from node to node in the system and leave the system from same node necessarily following the same order of nodes.

(iii) Customers may return to the nodes already visited, skip some nodes and even choose to remain in the same forever.

4) Define an open Jackson network ?

Suppose a queueing network consists of k nodes is called an open Jackson network, if it satisfies the following charecteristics.

(i) Customers arriving at node k from outside the system arrive in a poisson pattern with the average arrival rate r, and join the queue at i and wait for his turn for service.

(ii) Service times at the channels at node i are independent and each exponentially distributed with parameter μ.

(III) Once a customer gets the service completed at node i, he joins the queue at node j with probability Pij (whatever be the number of customers waiting at j for service), when i=1, 2,..., k and j=0, 1, 2, ..., k. Pio represents the probability that a customer leaves the system from node i after getting the service at i.

(iv) The utilization of all the queues is less than one.

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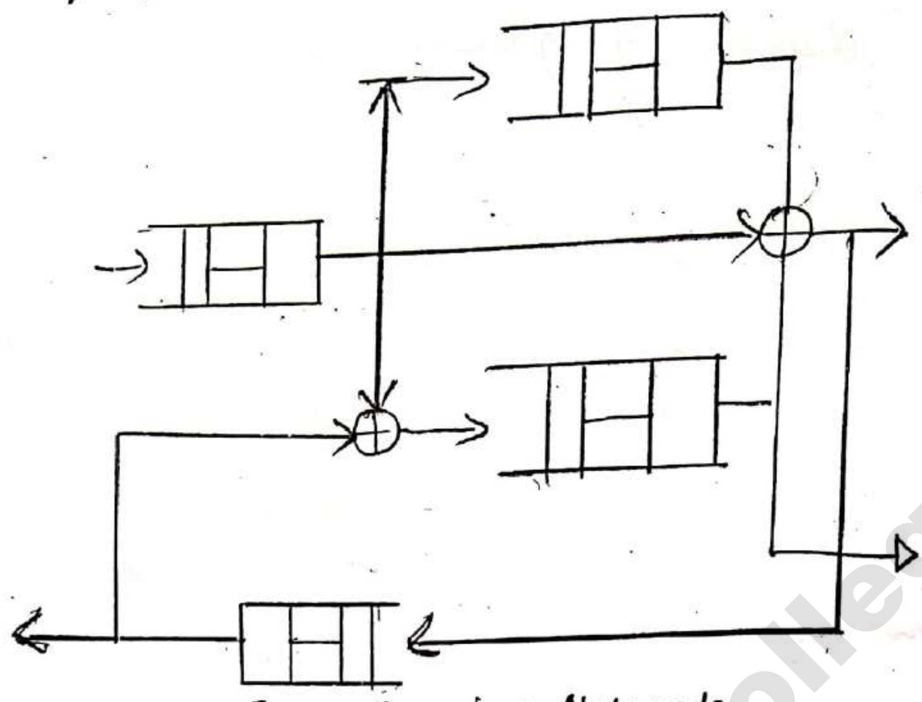
5) Define Open queueing network?

Open Network:

An open queuing network is charecterised by one or more sources of job arrivals and corresponding one or more sinks that absorb jobs departing from the network. If the network has multiple job classes then it must be open each class of jobs



Open Queueing Network

6) What do you mean by bottleneck of a network.

As the arrival rate λ in a 2-state tandem queue model increases, the node with the larger value of $\rho_i = \lambda / \mu_i$ will introduce instability. Hence the node with the large value of ρ_i is called the bottleneck of the system.

(or)

The service station for which the utilization factor is maximum among all the other service stations of the network is called the bottleneck of a network.

7) What do you mean by series queue with blocking?

This is a sequential queue model consisting of two service points S_1 and S_2 at each of which there is only one server and where no queue is allowed to form at either point

(50)

8) Write down the (flow balance) traffic equations for an open Jackson network.

Sol

Jackson's flow balance equations for this open model are

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i p_{ij}, \quad j = 1, 2, \ldots, k$$